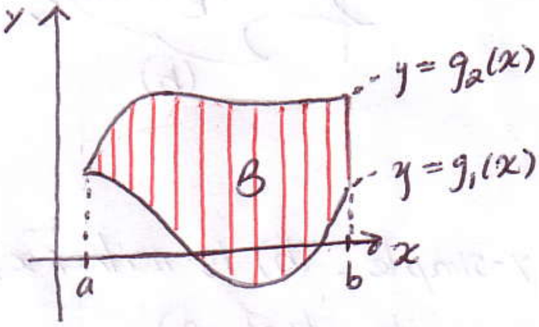
(5.3)

In the last section, we have defined the double-integral over a general region B. We have also seen how Cavalieri's principle can be applied to evaluate this integral in the special case when B is a rectangle. In this section, we extend the application of this principle to more general regions.

Def: A region $B \subset \mathbb{R}^2$ is y-simple if it is of the form

$$B = \{(x,y) : x \in [a,b],\ g_1(x) \leq y \leq g_2(x)\}$$

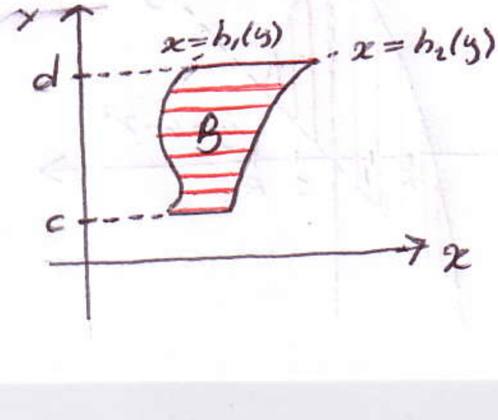
for constants a and b and functions $g_1, g_2 : [a,b] \to \mathbb{R}$



Similarly, a region B is x-simple if it is of the form

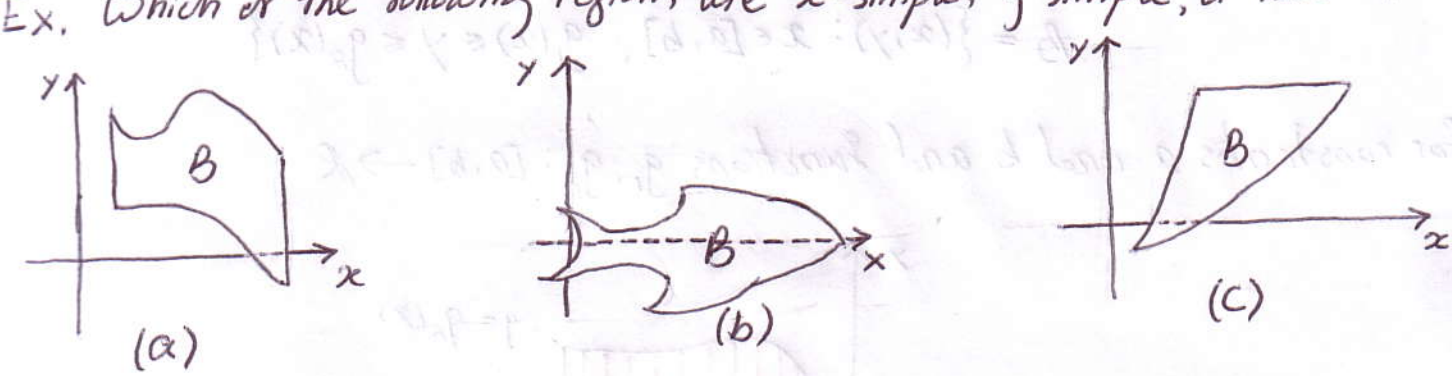
$$B = \{(x,y) : y \in [c,d],\ h_1(y) \leq x \leq h_2(y)\}$$

for constants c and d and functions $h_1, h_2 : [c,d] \to \mathbb{R}$.



x-simple and y-simple regions are easily recognized from their graphs. To identify a y-simple region, observe that the lower and upper curves between which the region is wedged are functions of x. These curves must therefore pass the vertical line test. By the same reasoning, x-simple regions are wedged between a left and a right curve. Since these curves are functions of y, they must pass the horizontal line test.

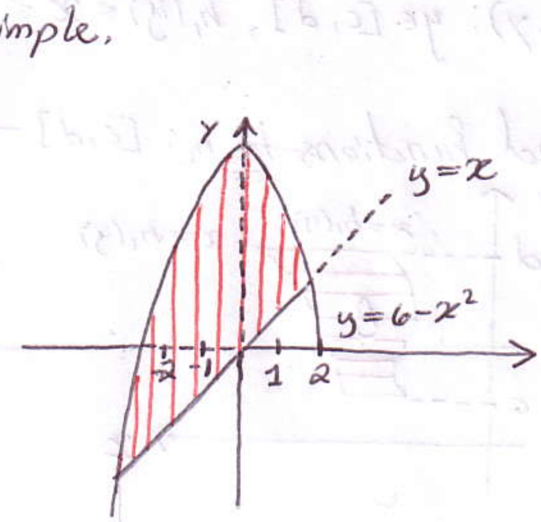
Ex. Which of the following regions are x-simple, y-simple, or neither.



Solution: (a) is y-simple, (b) is neither x, nor y-simple, and (c) is both x, and y-simple. (Why?)

Ex. Classify the region bounded by the curves $y=x$ and $y=6-x^2$ as either x, or y-simple.

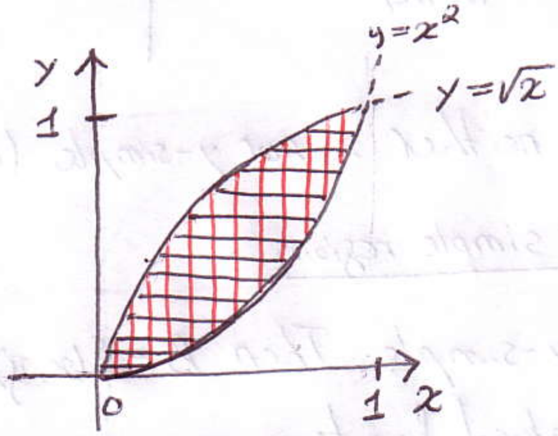
Solution:



The region is bounded by two functions of x. These two functions intersect when $x = 6 - x^2$ or $x^2 + x - 6 = (x+3)(x-2) = 0$. Thus, this region can be written as $\{(x,y) : x \in [-3,2],\ x \le y \le 6 - x^2\}$. In particular, this region is y-simple. Notice that the region is not x-simple.

Ex. Classify the region bounded by the curves $y = \sqrt{x}$ and $y = x^2$ as either x, or y-simple.

Solution:

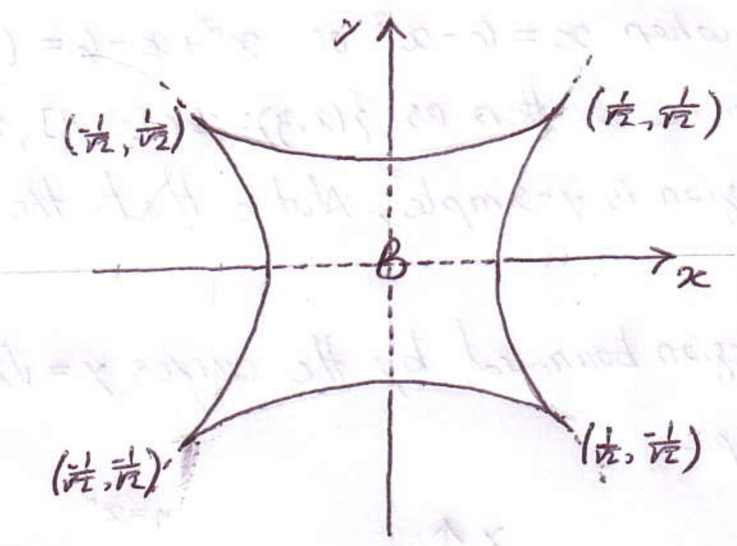


Observe that this region can be written as $\{(x,y) : x \in [0,1],\ x^2 \le y \le \sqrt{x}\}$. Thus, the region is y-simple.

On the other hand, since $x \ge 0$, $y = x^2 \Rightarrow x = \sqrt{y}$ and $y = \sqrt{x} \Rightarrow x = y^2$. In particular, this region can also be written as $\{(x,y) : y \in [0,1],\ y^2 \le x \le \sqrt{y}\}$. Hence this region is also x-simple. (Notice that this region passes the vertical and the horizontal line tests)

Ex. Let B be the region bounded by the circles $x^2 + (y - \sqrt{2})^2 = 1$, $(x + \sqrt{2})^2 + y^2 = 1$, $x^2 + (y + \sqrt{2})^2 = 1$, $(x - \sqrt{2})^2 + y^2 = 1$. Is B x-simple, y-simple, both, or neither?

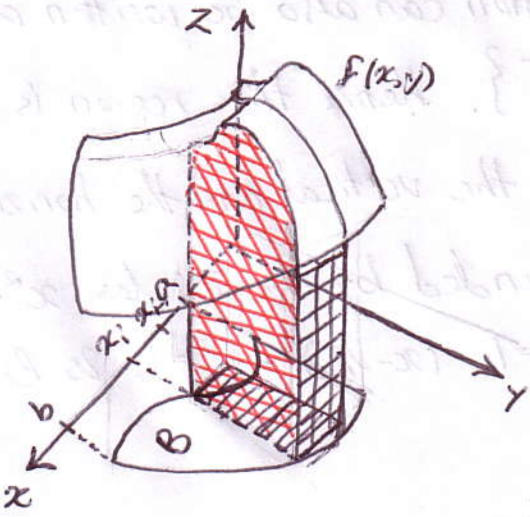
Solution:



This region is neither x, nor y-simple (why?).

Integrals over simple regions

Suppose B is y-simple. Then $B = \{(x,y) : x \in [a,b],\ g_1(x) \le y \le g_2(x)\}$ for some real-valued functions $g_1, g_2 : [a,b] \to \mathbb{R}$. Suppose also that $B \subseteq \mathcal{U} \subseteq \mathbb{R}^2$ and $f : \mathcal{U} \to \mathbb{R}$ is continuous on B. Then, we may use Cavalieri's principle to break the double-integral $\iint_B f$ into the iterated integral form $\int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$. To see this, consider the picture below;



Partition $[a,b]$ into n subintervals $a = x_0 < x_1 < \dots < x_{n-1} < x_n = b$.
By Cavalieri's principle, $\iint_B f = \lim_{n\to\infty} \sum_{i=1}^{n} A(x_i)\,\Delta x_i = \int_a^b A(x)\,dx$.

Notice that $A(x_i)$ is the area below the curve $f(x_i, y)$ and over the interval $[g_1(x_i), g_2(x_i)]$. In other words, $A(x_i) = \int_{g_1(x_i)}^{g_2(x_i)} f(x_i, y)\,dy$

Thus, $\iint_B f = \lim_{n\to\infty} \sum_{i=1}^{n} \left(\int_{g_1(x_i)}^{g_2(x_i)} f(x_i, y)\,dy \right) \Delta x_i =$

$= \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$ (Make sure that you have a thorough grasp of this argument.)

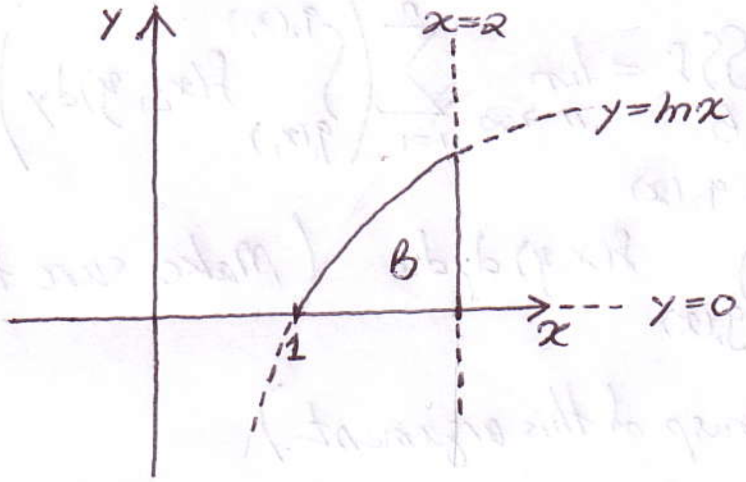
Similarly, if B is an x-simple region of the form $B = \{(x,y) : y \in [c,d],\ h_1(y) \le x \le h_2(y)\}$, $\iint_B f = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$

Ex. Let $B = \{(x,y) : x \in [1,2],\ 1-x^2 \le y \le 1+x^2\}$. Let $f(x,y) = xy$.
Calculate $\iint_B f$.

Solution: Observe that B is y-simple. Therefore $\iint_B f = \int_1^2 \int_{1-x^2}^{1+x^2} xy\,dy\,dx$.
Now, $\int_1^2 \int_{1-x^2}^{1+x^2} xy\,dy\,dx = \int_1^2 \left(\frac{1}{2} x y^2 \Big|_{y=1-x^2}^{y=1+x^2} \right) dx =$
$= \int_1^2 \frac{x}{2} \left((1+x^2)^2 - (1-x^2)^2 \right) dx = \int_1^2 2x^3\,dx = \frac{15}{2}$.

Ex. Let B be the region in the xy-plane bounded by the curve $y = \ln x$ and the lines $y = 0$ and $x = 2$. Find the volume of the solid T lying above B and beneath the surface $z = e^{y-x}$.

Solution:



The region shown above is both x- and y-simple. Treating it as y-simple, we describe it by the inequalities

$$1 \le x \le 2$$
$$0 \le y \le \ln x.$$

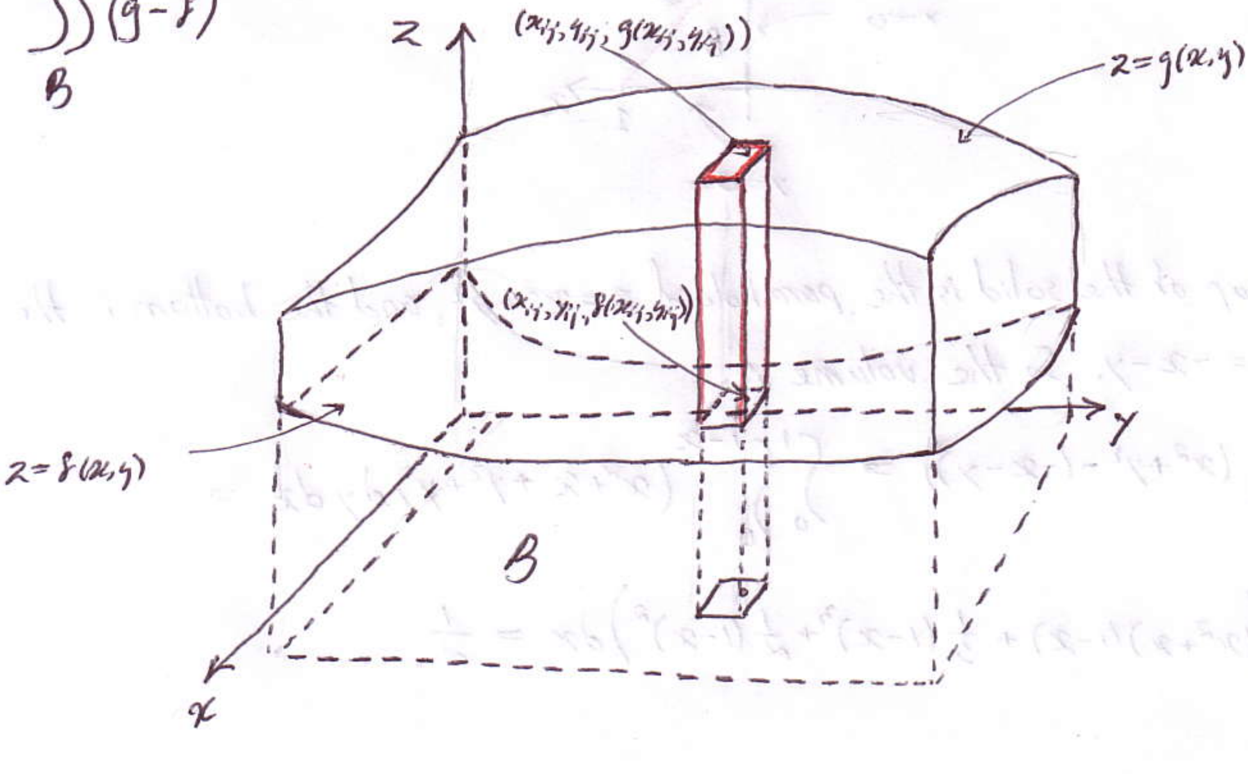
Since e^{y-x} is nonnegative, the double integral of e^{y-x} over B will represent the volume of T:

$$\iint_B e^{y-x} = \int_1^2 \int_0^{\ln x} e^{y-x}\,dy\,dx = \int_1^2 e^{y-x}\Big|_0^{\ln x} dx = \int_1^2 (e^{\ln x - x} - e^{-x})\,dx =$$
$$= \int_1^2 (x-1)e^{-x}\,dx = -(x-1)e^{-x}\Big|_1^2 - \int_1^2 -e^{-x}\,dx = e^{-1} - 2e^{-2}.$$

If f and g are two functions that satisfy $f(x,y) \leq g(x,y)$ for all (x,y) in a region B of the xy-plane, then it makes sense to ask for the volume of the solid T lying over B between the surfaces $z=f(x,y)$ and $z=g(x,y)$. In the picture below, we see that if we partition B into small rectangles with areas $\Delta x_i \Delta y_j$, choose a point (x_{ij}, y_{ij}) in each of these rectangles, and construct a box with height $g(x_{ij}, y_{ij}) - f(x_{ij}, y_{ij})$, then the sum of the volumes $\left(g(x_{ij}, y_{ij}) - f(x_{ij}, y_{ij})\right)\Delta x_i \Delta y_j$ of these boxes approximates the volume of the solid T. This leads us to define the volume of T as

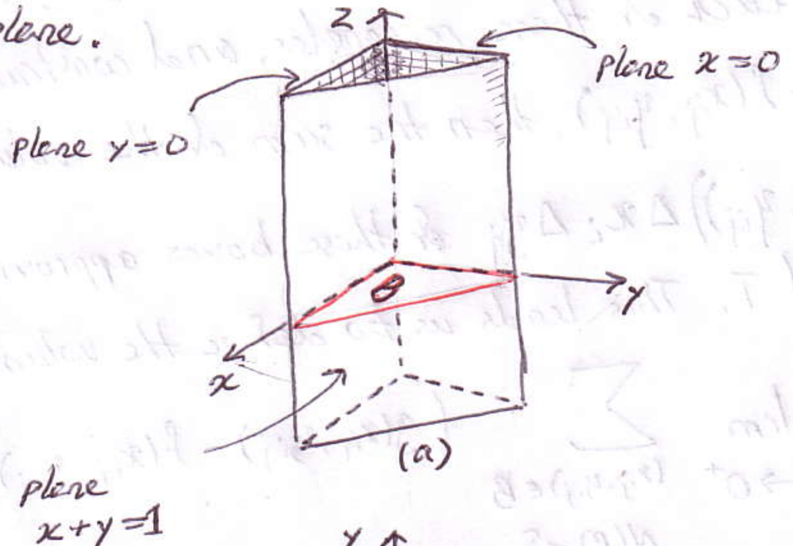
$$V(T) = \lim_{\delta \to 0^+} \sum_{\substack{(x_{ij}, y_{ij}) \in B \\ N(P) < \delta}} \left(g(x_{ij}, y_{ij}) - f(x_{ij}, y_{ij}) \right) \Delta x_i \Delta y_j =$$

$$= \iint_B (g-f)$$

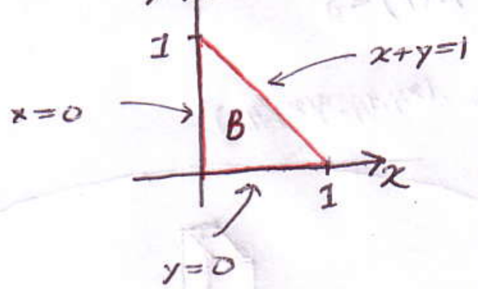


Ex. Find the volume of the solid in the first octant bounded by the planes $y=0$, $x=0$, $x+y=1$, $z=-x-y$, and the paraboloid $z=x^2+y^2$.

Solution: The three planes $x=0, y=0$, and $x+y=1$ form a wedge-shaped cylinder that cuts off the region $B=\{(x,y): x\in[0,1], 0\le y\le 1-x\}$ in the xy-plane.



(a)

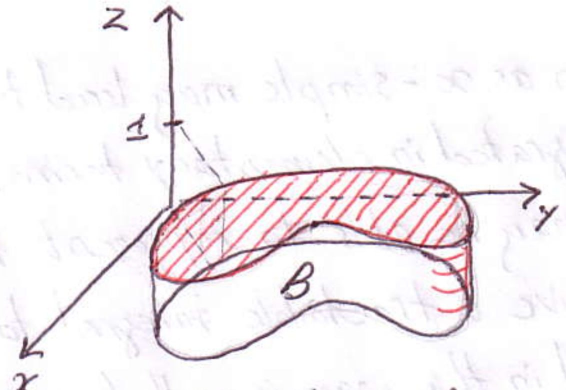


The top of the solid is the paraboloid $z=x^2+y^2$, and the bottom is the plane $z=-x-y$. So the volume is

$$\iint_B (x^2+y^2-(-x-y)) = \int_0^1\int_0^{1-x} (x^2+x+y^2+y)\,dy\,dx =$$

$$= \int_0^1\left((x^2+x)(1-x)+\frac{1}{3}(1-x)^3+\frac{1}{2}(1-x)^2\right)dx = \frac{1}{2}.$$

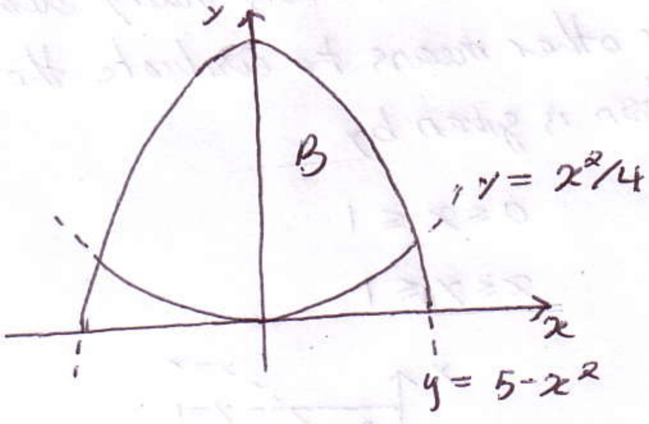
Double integrals can also represent areas. Given a region B, we can place a cylinder, whose base is of the shape B and whose height is one unit, on that region. The volume of this cylinder is equal to the area of B (Why?). In particular, $A(B) = \iint_B 1$.



Cylinder with base B & height 1.

Ex. Find the area of the region bounded by the parabolas $y = x^2/4$ and $y = 5 - x^2$.

Solution:



The curves intersect when $5 - x^2 = x^2/4$. This gives $x = \pm 2$, so the region is y-simple and described by the inequalities

$$-2 \le x \le 2$$
$$\frac{x^2}{4} \le y \le 5 - x^2.$$

Thus the area is

$$\int_{-2}^{2}\int_{x^2/4}^{5-x^2} dy\,dx = \int_{-2}^{2}\left(5-\tfrac{5}{4}x^2\right)dx = \frac{40}{3}$$

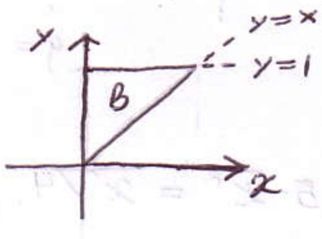
Changing the order of integration

Occasionally, treating a region as x-simple may lead to an integral that cannot be integrated in elementary terms. However, if the region is y-simple, we may be able to represent the double integral in a way that does give a tractable integral to evaluate. Rewriting an iterated integral in this way is called reversing the order of integration.

Ex. Evaluate $\int_0^1\int_x^1 e^{y^2}\,dy\,dx$.

Solution: The function e^{y^2} does not have an antiderivative that can be represented in terms of finitely many elementary functions, so we must look for other means to evaluate the integral. The region of integration is given by

$$0 \le x \le 1$$
$$x \le y \le 1$$



This region is also x-simple and described by

$$0 \le y \le 1$$
$$0 \le x \le y$$

Thus

$$\int_0^1\int_x^1 e^{y^2}\,dy\,dx = \int_0^1\int_0^y e^{y^2}\,dx\,dy = \int_0^1 x e^{y^2}\Big|_{x=0}^{x=y} dy =$$

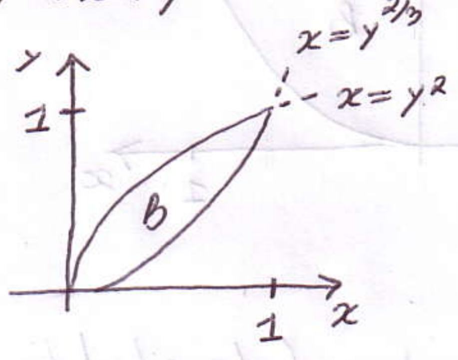
$$= \int_0^1 y e^{y^2} dy = \frac{1}{2} e^{y^2}\Big|_0^1 = \frac{1}{2}(e-1)$$

Ex. Evaluate $\displaystyle\int_0^1\int_{y^2}^{y^{2/3}} \frac{x^{3/2}\cos x}{y^2}\,dx\,dy$

Solution: Again, this iterated integral is problematic. The function $\dfrac{x^{3/2}\cos x}{y^2}$ does not have an elementary antiderivative with respect to x. Let's see what happens if we reverse the order of integration:

The region of integration is given by

$$0 \le y \le 1$$
$$y^2 \le x \le y^{2/3}$$



This region is also y-simple and described by

$$0 \le x \le 1$$
$$x^{3/2} \le y \le x^{1/2}$$

Thus

$$\int_0^1 \int_{y^2}^{y^{2/3}} \frac{x^{3/2}\cos x}{y^2}\,dx\,dy = \int_0^1 \int_{x^{3/2}}^{x^{1/2}} \frac{x^{3/2}\cos x}{y^2}\,dy\,dx =$$

$$= \int_0^1 \frac{-x^{3/2}\cos x}{y}\Big|_{y=x^{3/2}}^{y=x^{1/2}} dx = \int_0^1 (-x\cos x + \cos x)\,dx =$$

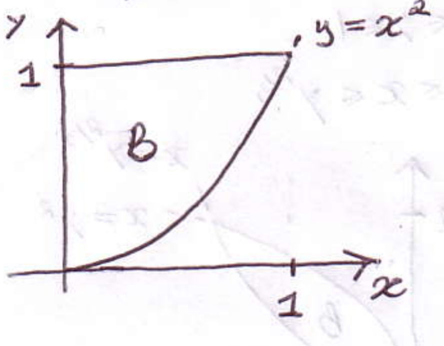
$$= (-x\sin x - \cos x + \sin x)\Big|_0^1 = -\sin(1) - \cos(1) + \sin(1) + 1 =$$

$$= 1 - \sin(1).$$

Ex. Evaluate $\int_0^1 \int_{x^2}^1 x e^{y^2}\,dy\,dx$

Solution: The desired region is

$$0 \le x \le 1$$
$$x^2 \le y \le 1$$



This region is also x-simple and described by

$$0 \le y \le 1$$
$$0 \le x \le \sqrt{y}$$

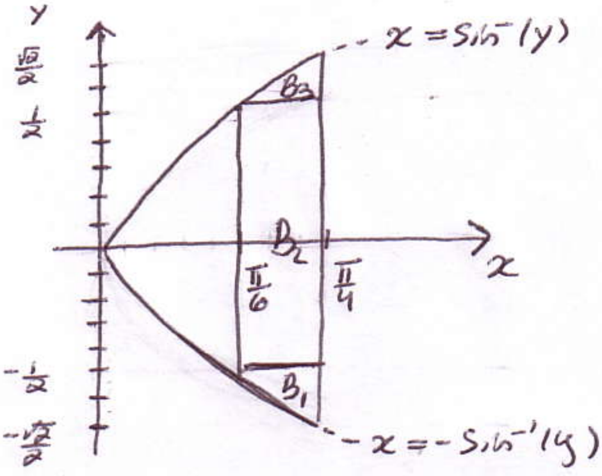
Thus $$\int_0^1 \int_{x^2}^1 x e^{y^2}\,dy\,dx = \int_0^1 \int_0^{\sqrt{y}} x e^{y^2}\,dx\,dy = \int_0^1 \frac{x^2}{2} e^{y^2}\Big|_{x=0}^{x=\sqrt{y}} dy =$$

$$= \int_0^1 \frac{y}{2} e^{y^2}\,dy = \frac{1}{4}\int_0^1 2y e^{y^2}\,dy = \frac{1}{4}\left(e^{y^2}\right)\Big|_0^1 = \frac{1}{4}(e-1)$$

Ex. Evaluate $\int_{-\frac{\sqrt{2}}{2}}^{-\frac{1}{2}}\int_{-\sin^{-1}(y)}^{\frac{\pi}{4}} \frac{3y^2\ln(\cos^2 x)}{\sin x \sin 2x}\,dx\,dy +$

$+ \int_{-\frac{1}{2}}^{\frac{1}{2}}\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{3y^2\ln(\cos^2 x)}{\sin x \sin 2x}\,dx\,dy + \int_{\frac{1}{2}}^{\frac{\sqrt{2}}{2}}\int_{\sin^{-1}(y)}^{\frac{\pi}{4}} \frac{3y^2\ln(\cos^2 x)}{\sin x \sin 2x}\,dx\,dy$

Solution: The desired region is depicted in the picture below.



Notice that $B = B_1 \cup B_2 \cup B_3$ is y-simple and described by

$$\frac{\pi}{6} \le x \le \frac{\pi}{4}$$

$$-\sin x \le y \le \sin x$$

Thus the sum $\sum_{i=1}^{3}\iint_{B_i} \frac{3y^2\ln(\cos^2 x)}{\sin x \sin 2x}$ can be reduced to a single integral

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{4}}\int_{-\sin x}^{\sin x} \frac{3y^2\ln(\cos^2 x)}{\sin x \sin 2x}\,dy\,dx = \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{2\sin^3 x\ln(\cos^2 x)}{2\sin^2 x \cos x}\,dx =$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} 2\tan x \ln(\cos x)\,dx = -\left(\ln(\cos x)\right)^2 \Big|_{\frac{\pi}{6}}^{\frac{\pi}{4}} = -\left(\ln\left(\tfrac{\sqrt{2}}{2}\right)\right)^2 + \left(\ln\left(\tfrac{\sqrt{3}}{2}\right)\right)^2$$