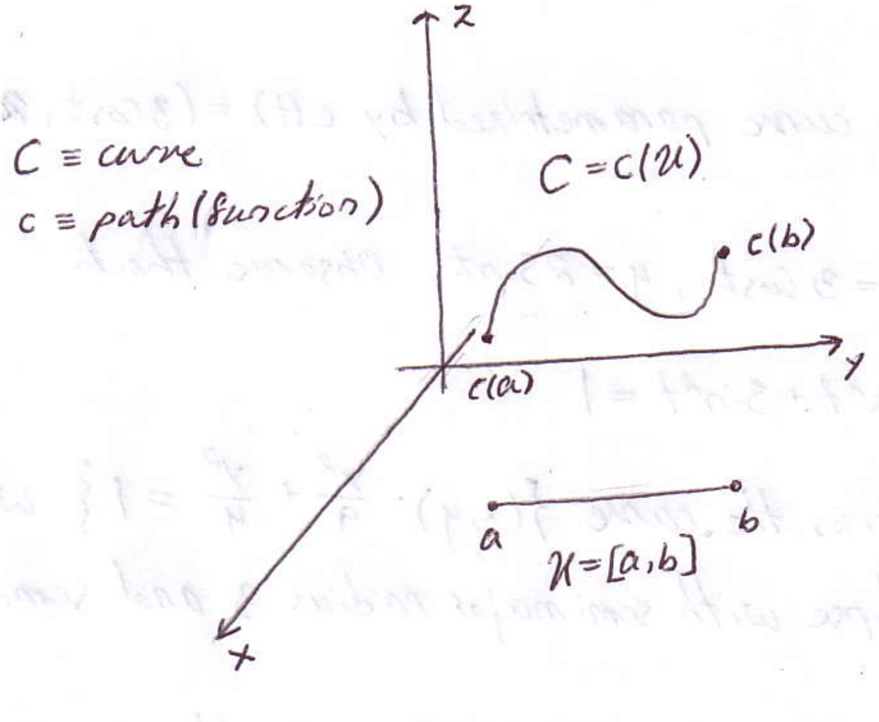
(2.4)

In earlier sections we have seen how a particle traveling through space with a fixed direction determines a line. The set of all points on this line consists of points that are visited by the particle at some moment in time $t \in \mathbb{R}$.
More generally, we can think of a curve C in space as the set of points visited by a particle whose position at a given moment of time $t \in \mathbb{R}$ is $c(t)$, where $c: \mathcal{U} \subset \mathbb{R} \to \mathbb{R}^3$ is a function s.t. $c(\mathcal{U}) = C$. We call this function c a path and say that c parametrizes the curve C.

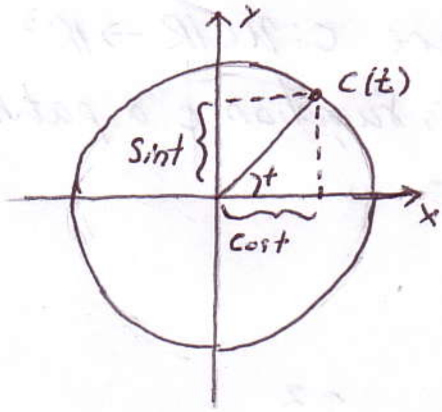


Ex. Let $c:(0,\infty)\subset \mathbb{R}\to\mathbb{R}^3$ be defined by

$$c(t)=(3t+1,-5t+2,t)$$

Then $C=c(0,\infty)$ which is a ray from the point $(1,2,0)$ in the direction of $\vec{v}=3i-5j+k$.

Ex. The unit circle is the curve $C=c([0,2\pi])$ where $c:[0,2\pi]\to\mathbb{R}^2$ is given by $c(t)=(\cos t,\sin t)$



Ex. Describe the curve parametrized by $c(t)=(3\cos t, 2\sin t)$ $t\in[0,2\pi]$

Solution: Let $x=3\cos t$, $y=2\sin t$. Observe that

$$\left(\frac{x}{3}\right)^2+\left(\frac{y}{2}\right)^2=\cos^2 t+\sin^2 t=1$$

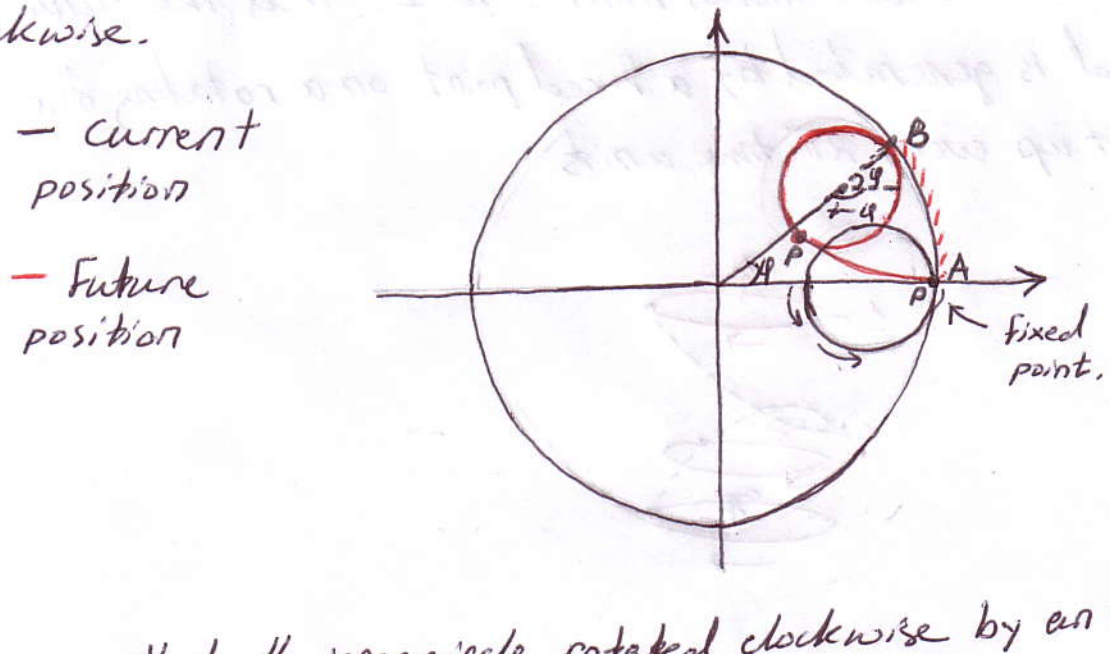
Hence c parametrizes the curve $\{(x,y):\frac{x^2}{9}+\frac{y^2}{4}=1\}$ which is an ellipse with semimajor radius 3 and semiminor radius 2.

Notice that $f(t)=(3\cos 7t, 2\sin 7t)$ is another parametrization of the same ellipse. Thus, just like with lines, parametric descriptions of curves are not unique.

Many planar curves of historical interest have definitions that make them more easily represented by a parametrization than by a single equation in x and y. One such curve is the hypocycloid, the path traced by a point on the circumference of a circle that rolls along the inside of a larger circle.

Ex. Find a parametrization of the hypocycloid traced by a point on the circumference of a circle of radius r that rolls along inside another circle of larger radius R.

Solution: We need to determine the location of a point, fixed on the rotating circle of radius r as a function of time. First, assume that the inner circle rotates with constant angular speed of 1 rad/min clockwise.



Suppose that the inner circle rotated clockwise by an angle t. Before the rotation, the fixed point p coinsided with point A on the large circle. Now, the inner circle touches the bigger one at point B. Observe that the red arc $\widehat{PB}$ has the same length as the arc $\widehat{AB}$. Thus length of $\widehat{AB} = R\varphi =$ length of $\widehat{PB} = rt$. Hence $\varphi = \frac{r}{R} t$

The new position of p is consequently the sum of the vectors $\left((R-r)\cos\varphi, (R-r)\sin\varphi\right) + \left(r\cos(\varphi-t), r\sin(\varphi-t)\right) =$

$$= \left((R-r)\cos(\tfrac{r}{R}t), (R-r)\sin(\tfrac{r}{R}t)\right) + \left(r\cos(\tfrac{r}{R}t-t), r\sin(\tfrac{r}{R}t-t)\right) =$$

$$= \left((R-r)\cos(\tfrac{r}{R}t) + r\cos(\tfrac{r-R}{R}t), (R-r)\sin(\tfrac{r}{R}t) + r\sin(\tfrac{r-R}{R}t)\right)$$

$$= \left((R-r)\cos(\tfrac{r}{R}t) + r\cos(\tfrac{R-r}{R}t), (R-r)\sin(\tfrac{r}{R}t) - r\sin(\tfrac{R-r}{R}t)\right)$$

Ex. Plot the curve parametrized by

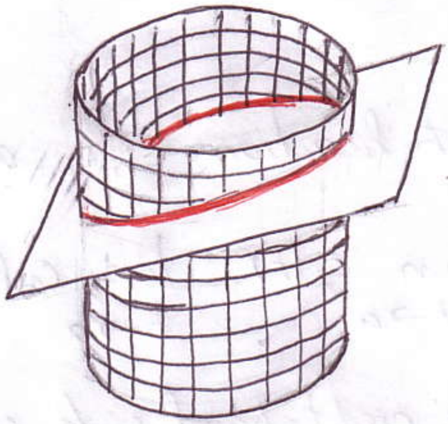
$$c(t) = \left(\cos t, \sin t, \frac{t}{2\pi}\right)$$

Solution: As t varies from 0 to 2π the x, y coordinates trace out a circle while the z coordinate moves from 0 to 1. Thus, the curve that is being traced is generated by a fixed point on a rotating disc that moves 1 unit up every 2π time units.

The curve is consequently a helix.

Ex. Find a parametrization for the curve in which the cylinder $x^2+y^2=9$ and the plane $2x+3y-z=1$ intersect.

Solution:



The cylinder $C=\{(x,y,z): x^2+y^2=9\}$ intersects the plane $P=\{(x,y,z): 2x+3y-z=1\} = \{(x,y,2x+3y-1): x,y\in\mathbb{R}\}$.

This intersection is the set $C\cap P = \{(x,y,2x+3y-1): x^2+y^2=9\}$

Let $x=3\cos t$, $y=3\sin t$. Then $z=2x+3y-1 = 6\cos t+9\sin t-1$

Consequently, the set $C\cap P$ is parametrized by the path

$c(t) = (3\cos t, 3\sin t, 6\cos t+9\sin t-1) \quad t\in[0,2\pi]$

Derivatives and motion

As a particle is traveling through a path, its velocity and accelleration varies. In this section we are concerned with developing a mathematical model that will allow us to recover the information about a particle's velocity and acceleration from its position and visa versa.

Def: $\lim_{t \to a} c(t) = L$ if for any choice of $\epsilon > 0$ there exists a positive number δ s.t. $\|c(t) - L\| < \epsilon$. whenever $0 < |t-a| < \delta$.
If, in addition, $L = c(a)$, then we say that c is continuous at a.

Thm: If c has component functions c_1, c_2, and c_3, then

$$\lim_{t \to a} c(t) = \left(\lim_{t \to a} c_1(t), \lim_{t \to a} c_2(t), \lim_{t \to a} c_3(t) \right)$$

iff the three limits on the right-hand side exist.

Proof: Suppose $\lim_{t \to a} c(t)$ exists; call it $L = (L_1, L_2, L_3)$.

Then by the definition above, for any $\epsilon > 0$ there is a $\delta > 0$ s.t.

$$\sqrt{(c_1(t) - L_1)^2 + (c_2(t) - L_2)^2 + (c_3(t) - L_3)^2} < \epsilon \qquad (1)$$

whenever $0 < |t-a| < \delta$.
The left-hand side of (1) is bigger than $|c_i(t) - L_i|$ for $i \in \{1,2,3\}$
It follows that

$$|c_1(t) - L_1| < \epsilon$$

$$|c_2(t) - L_2| < \epsilon$$

$$|c_3(t) - L_3| < \epsilon$$

whenever $0 < |t-a| < \delta$.

To prove the other direction, suppose that $\lim_{t \to a} c_i(t) = L_i$ for $i \in \{1,2,3\}$. Then there exist $\delta_1, \delta_2, \delta_3$ s.t.

$$|c_i(t) - L_i| < \tfrac{\epsilon}{3} \text{ whenever } 0 < |t-a| < \delta_i$$

Let $\delta = \min\{\delta_1, \delta_2, \delta_3\}$, then

$$\sqrt{(c_1(t)-L_1)^2 + (c_2(t)-L_2)^2 + (c_3(t)-L_3)^2} \overset{\text{(by triangle ineq.)}}{\leq}$$

$$\leq |c_1(t) - L_1| + |c_2(t) - L_2| + |c_3(t) - L_3| < \tfrac{\epsilon}{3} + \tfrac{\epsilon}{3} + \tfrac{\epsilon}{3} = \epsilon.$$

Thus the existance of the limits of the component functions implies the existance of the limit of the path function

Ex. Let $f(t) = \left(\frac{e^t - 1}{t}, \frac{\cos t}{t+1}, t\sin(1/t)\right)$. Find $\lim_{t \to 0} f(t)$.

Solution: Let $f_1(t) = \frac{e^t-1}{t}$, $f_2(t) = \frac{\cos t}{t+1}$, $f_3(t) = t\sin(1/t)$.

Observe that $\lim_{t \to 0} f_1(t) = \lim_{t \to 0} \frac{e^{0+t} - e^0}{t} = \frac{d}{dx}(e^x)\Big|_{x=0} = e^0 = 1$

Similarly $\lim_{t \to 0} \frac{\cos t}{t+1} = 1$ and

$\lim_{t \to 0} t\sin(1/t) = 0$ because, by the squeeze theorem,

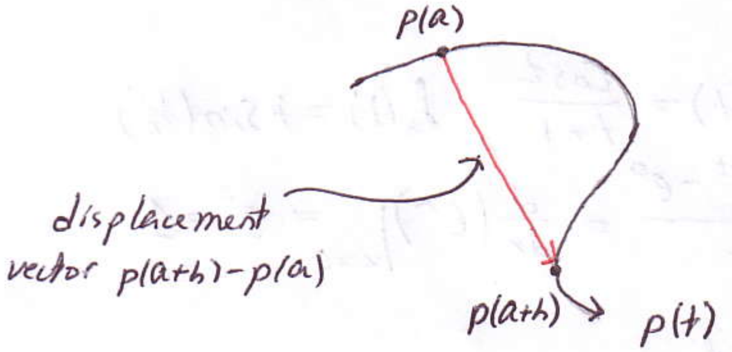
$-t \leq t\sin(1/t) \leq t$ from which it follows that

$$\lim_{t \to 0} -t \leq \lim_{t \to 0} t\sin(1/t) \leq \lim_{t \to 0} t$$

It follows that $\lim_{t\to 0} f(t)$ exists and $\lim_{t\to 0} f(t) = (1,1,0)$

Suppose that a particle travels in such a way that its position at time t is $p(t)$. What is its velocity?

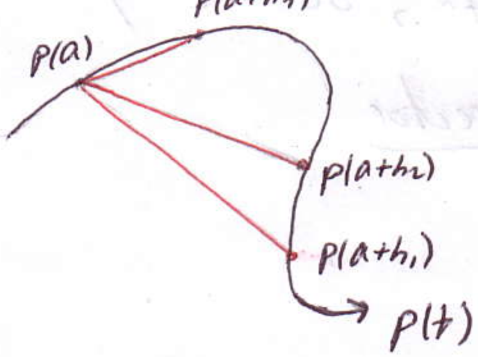
To answer this question, recall that velocity is measuring how quickly the position of this particle is changing. More precisely, suppose the particle is in position $p(a)$ at time $t=a$. After an increment of time h, the particle will be in position $p(a+h)$. The average velocity is the vector $\frac{p(a+h)-p(a)}{h}$. It measures how fast the particle should have traveled along the shortest path (i.e. a line) from position $p(a)$ to position $p(a+h)$ with constant speed to arrive at $p(a+h)$ in h time units:



In particular, if we only know that the particle visited position $p(a)$ at time a and position $p(a+h)$ at time $t=a+h$ (that is the position function p is unknown for arbitrary t), we may suppose that the particle traveled along the line $L(t) = p(a) + t\,\frac{p(a+h)-p(a)}{h}$

Observe that $L(0) = p(a)$ and $L(h) = p(a+h)$.
The line $L(t)$ is more closely approximating the actual trajectory of the particle, $p(t)$, for $t \in [a, a+h]$. This allows us to define the instantenious velocity of the particle at time $t=a$ to be
$\lim_{h \to 0} \frac{p(a+h)-p(a)}{h} = p'(a)$ and the corresponding tangent line

$L(t) = p(a) + t p'(a)$.



If $h_1 > h_2 > h_3$, it is reasonable to assume that $L_3(t)$ is a better approximation to the path $p(t)$

Thm: Let p be a path function with component functions p_1, p_2, p_3. If the component functions are differentiable then p is differentiable with derivative $p'(t) = (p_1'(t), p_2'(t), p_3'(t))$

Proof: $p'(t) = \lim_{h \to 0} \frac{p(t+h)-p(t)}{h} = \frac{1}{h}(p_1(t+h) - p_1(t), p_2(t+h) - p_2(t),$
$p_3(t+h) - p_3(t)) = \lim_{h \to 0} \left(\frac{p_1(t+h) - p_1(t)}{h}, \frac{p_2(t+h) - p_2(t)}{h}, \frac{p_3(t+h) - p_3(t)}{h} \right) =$

$$= \left(\lim_{h\to 0} \frac{P_1(t+h)-P_1(t)}{h}, \lim_{h\to 0} \frac{P_2(t+h)-P_2(t)}{h}, \lim_{h\to 0} \frac{P_3(t+h)-P_3(t)}{h} \right)$$

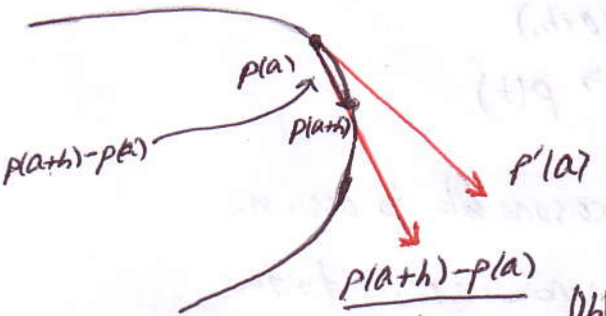
$$= \left(P_1'(t), P_2'(t), P_3'(t) \right)$$

Ex. Calculate the derivative of the function f given by

$$f(t) = \left((t^2+2t+5), (\ln t + \tfrac{1}{t}), (e^{3t}-e^{-t}) \right)$$

Solution: $f'(t) = \left(2t+2, \tfrac{1}{t} - \tfrac{1}{t^2}, 3e^{3t}+e^{-t} \right)$

Geometry of the derivative vector



$\frac{p(a+h)-p(a)}{h}$ $(|h|<1)$

The derivative $p'(a)$ is tangent to the path traced by p at the point $p(a)$.

Physical interpretation of the derivative vector

If we think of $p(t)$ as the position of an object at time t, then we refer to $v(t) = p'(t)$ as the velocity at time t. The magnitude $\|v(t)\|$ of the velocity is the object's speed.

Ex. Suppose that an object is moving through space along the path $c(t) = (t\sin t, \cos t, \sin t)$ and then at the moment when $t = \frac{\pi}{4}$ it suddenly begins traveling in the direction of its velocity vector with $\|v(\frac{\pi}{4})\|$ as its constant speed. What will be the object's position when $t = \frac{\pi}{2}$?

Solution: $v(\frac{\pi}{4}) = c'(t)\Big|_{t=\frac{\pi}{4}} = (\sin t + t\cos t, -\sin t, \cos t)\Big|_{t=\frac{\pi}{4}} =$

$$= \left(\frac{\sqrt{2}}{2} + \frac{\pi\sqrt{2}}{8}, -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) = \frac{\sqrt{2}}{2}\left(1 + \frac{\pi}{4}, -1, 1\right)$$

The object's path from $t = \frac{\pi}{4}$ onwards is a line $L(t) = \vec{a} + t\vec{m}$ where $L(\frac{\pi}{4}) = c(\frac{\pi}{4}) = (\frac{\pi}{4}, 1, 1)\frac{\sqrt{2}}{2}$ and $\vec{m} = v(\frac{\pi}{4})$.

Thus $L(t) = c(\frac{\pi}{4}) + (t - \frac{\pi}{4})v(\frac{\pi}{4}) = \frac{\sqrt{2}}{2}(\frac{\pi}{4}, 1, 1) + (t - \frac{\pi}{4})\frac{\sqrt{2}}{2}(1 + \frac{\pi}{4}, -1, 1$

The object's position at $t = \frac{\pi}{2}$ is $L(\frac{\pi}{2}) = \frac{\sqrt{2}}{2}(\frac{\pi}{4}, 1, 1) + \frac{\pi}{4}\frac{\sqrt{2}}{2}(1 + \frac{\pi}{4}, -1, 1)$

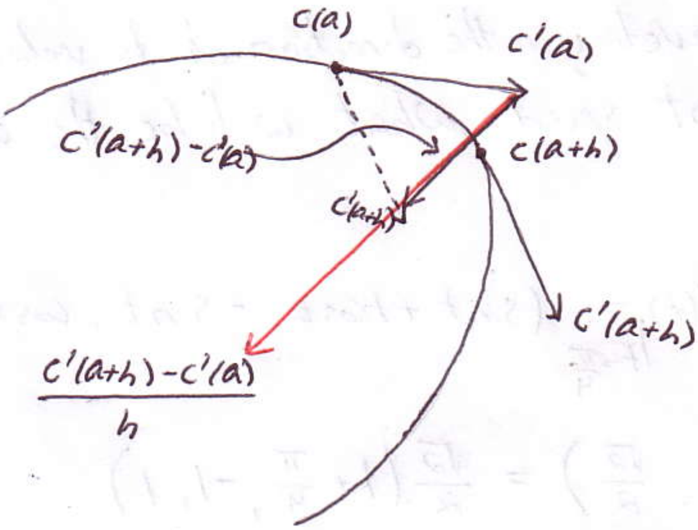
The second derivative and accelleration

Let $c(t)$ be a parametrization of a curve C. Then we define

$$c''(a) = \lim_{h \to 0} \frac{c'(a+h) - c'(a)}{h}.$$

Thus $c''(a)$ is a vector. The vector $\frac{c'(a+h) - c'(a)}{h}$ may be viewed as the average change in velocity over an increment of time h. $c''(a)$ may therefore be thought of as the acceleration of the particle at time $t = a$.

The figure below helps to visualize the relationship between $c(t)$ and $c''(a)$.



The acceleration vector $c''(a)$ points toward the concave side of the curve traced by c.

Vector-valued functions obey differentiation rules that are analogous to those for scaler-valued functions. This is described in the theorem below. (You should supply the proof)

Thm: let f and g be differentiable vector-valued functions of one variable, and let h be a differentiable scaler-valued function. Then the following identities hold.

1. $\frac{d}{dt}(f \pm g) = f' \pm g'$
2. $\frac{d}{dt}(cf) = cf'$ for $c \in \mathbb{R}$
3. $\frac{d}{dt}(hf) = h'f + hf'$
4. $\frac{d}{dt}(f \cdot g) = f' \cdot g + f \cdot g'$
5. $\frac{d}{dt}(f \times g) = f' \times g + f \times g'$
6. $\frac{d}{dt} f(h(t)) = h'(t) f'(h(t))$

Since the derivative of a vector-valued function is the vector with components the derivatives of the function's components, the most general antiderivative of a function $f=(f_1,f_2,f_3)$ is the vector with components the general antiderivatives of f_1, f_2, and f_3.

Def: For a continuous function $f=(f_1,f_2,f_3)$,

$$\int f(t)\,dt = \left(\int f_1(t)\,dt, \int f_2(t)\,dt, \int f_3(t)\,dt\right)$$

Ex. Find the position of a particle whose position is $(1,2,-3)$ when $t=0$ and whose velocity is $v(t)=(e^t, 2+e^t, e^{2t})$

Solution: $f(t)=\int v(t)\,dt = \left(\int e^t dt, \int (2+e^t)dt, \int e^{2t}dt\right) =$

$= \left(e^t+c_1,\ 2t+e^t+c_2,\ \frac{1}{2}e^{2t}+c_3\right)$.

When $t=0$ this gives $\left(1+c_1, 1+c_2, \frac{1}{2}+c_3\right)=(1,2,-3)$

$\Rightarrow c_1=0,\ c_2=1,\ c_3=-3-\frac{1}{2}=-\frac{7}{2}$.

Thus $f(t)=\left(e^t, 2t+e^t+1, \frac{1}{2}e^{2t}-\frac{7}{2}\right)$

Ex. A particle moves so that its position vector has constant magnitude. Show that its position vector is always perpendicular to its velocity.

Solution: Let $p(t)$ be the position of the particle at time t. By hypothesis $\|p(t)\|=c\in\mathbb{R}$. This implies that $\|p(t)\|^2=c^2$

Recall that $\|p(t)\|^2 = p(t)\cdot p(t)$. Now $\frac{d}{dt}\left(\|p(t)\|^2\right) = \frac{d}{dt}(c^2) = 0$

Thus $0 = \frac{d}{dt}(p(t)\cdot p(t)) = \frac{d}{dt}(p(t))\cdot p(t) + \frac{d}{dt}(p(t))\cdot p(t) =$

$= 2p'(t)\cdot p(t) \Rightarrow 0 = p'(t)\cdot p(t) \Rightarrow 0 = v(t)\cdot p(t).$

Hence position and velocity are perpendicular to each other.

Applications in Physics (optional)

Ex. A puck of mass m slides in a circle of radius r on a frictionless table while attached to a hanging cylinder of mass M by a cord through a hole in the table. What speed keeps the cylinder at rest?

Solution:

Since the puck moves in a circle, its position $p(t) = (r\cos\omega t, r\sin\omega t)$ where $\omega \in \mathbb{R}$. To keep the cylinder in equalibrium, the circular motion must generate a force equal in magnitude to Mg, where g is acceleration due to gravity. The acceleration of the puck is given by $p''(t) = -r\omega^2(\cos\omega t, \sin\omega t)$

Thus, the force on the puck, $F(t) = mp''(t) = -mr\omega^2(\cos\omega t, \sin\omega t)$

The magnitude of $F(t)$, $\|F(t)\| = mr\omega^2$, must be equal to the weight of the cylinder Mg.

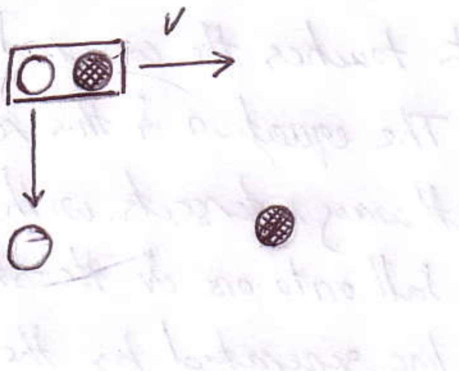
In particular, $mr\omega^2 = Mg \Rightarrow \omega = \sqrt{\frac{Mg}{mr}}$.

Observe that the speed of the puck is $\|p'(t)\| = r\omega$

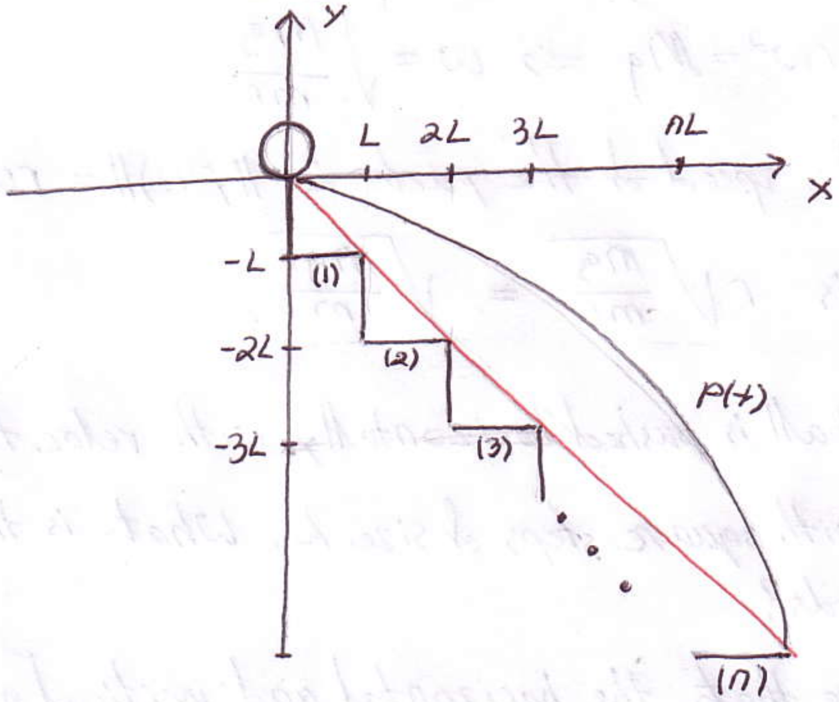
Thus the speed is $r\sqrt{\frac{Mg}{mr}} = \sqrt{\frac{Mgr}{m}}$.

Ex. Suppose a ball is pushed horizontally with velocity v from a staircase with square steps of size h. What is the step on which the ball lands?

Solution: Observe that the horizontal and vertical motions of the ball are independent. In particular, if one ball is dropped down and another ball is simultaneously launched horizontally with velocity v, the two balls will hit the ground at the same time.



The ball position function satisfies the equations $p''(t) = (0, -g)$, $p'(0) = (v, 0)$ and $p(0) = (0,0)$ as seen in the figure below:



It follows that $p(t) = (vt, -\frac{1}{2}gt^2)$ for all t for which the ball is airborne. To find the step on which the ball lands, suppose there is a lazer beam that touches the corner of each step (drawn in red in the figure above). The equation of this red line is $y = -x$. Notice that the ball's pathway intersects with the lazer when $t=0$. Since the ball will have to fall onto one of the steps, say the n^{th} step, it will have to intersect the line generated by the lazer beam yet again. When this happens there could be no further intersections, because if t is a moment when the ball intersects the beam $p(t) = (vt, -\frac{1}{2}gt^2) \equiv (x, -x)$ which means that this time t is a solution to the equation $vt = \frac{1}{2}gt^2$ which is a quadratic equation with at most 2 solutions.

Thus, if $t \neq 0$, $vt = \frac{1}{2}gt^2 \Rightarrow t = 2v/g$. Because the ball cannot cross the line $y = -x$ again, it will fall on the step directly below the x position $vt = v\left(\frac{2v}{g}\right) = \frac{2v^2}{g}$.

In particular, the ball falls onto the n^{th} step if

$$(n-1)L < \frac{2v^2}{g} \leq nL \qquad \text{or}$$

$$n - 1 < \frac{2v^2}{gL} \leq n$$

Hence, the step number, on which the ball falls, given that it is launched with horizontal velocity v from a staircase with square steps of length L and subject to constant acceleration g is given by the function

$N(v, L, g) = \left\lceil \frac{2v^2}{gL} \right\rceil$ where $\lceil \cdot \rceil$ is the ceiling function.

For example $\lceil 3.1 \rceil = \lceil 3.5 \rceil = \lceil 3.999 \rceil = 4$.