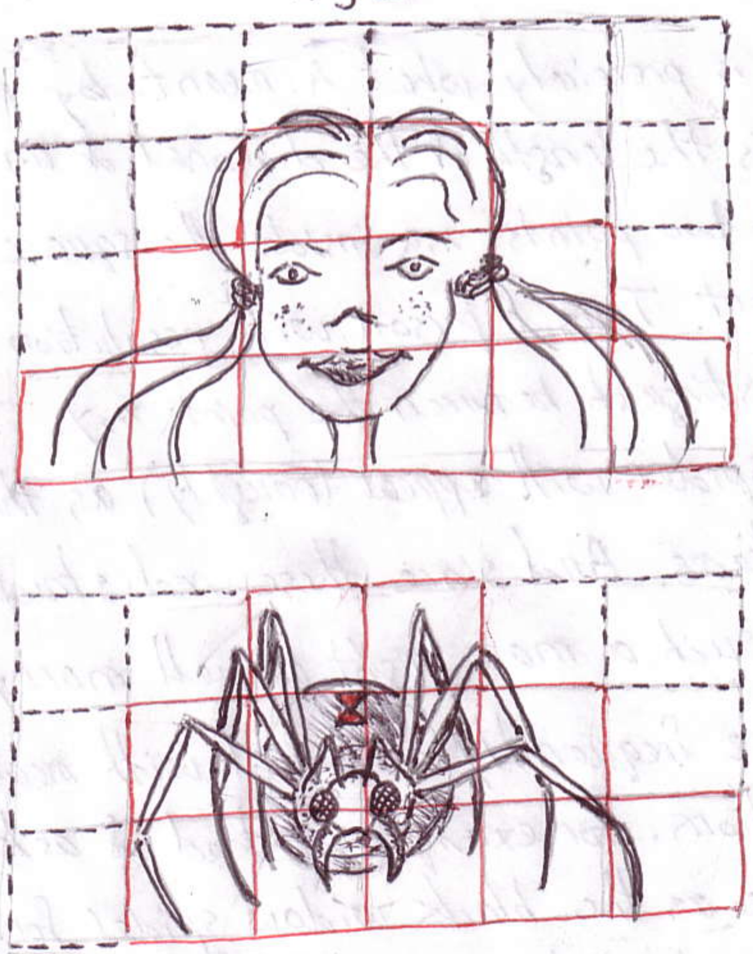
(2.2)

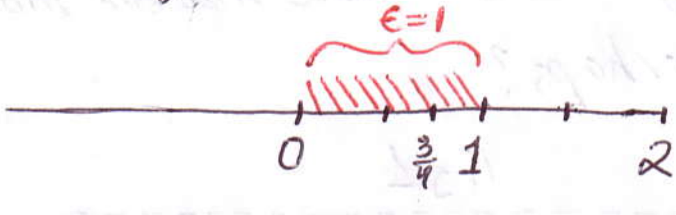
So you meet this great girl in a pub. Well, the pub is dark and, let's face it, you probably had, as Germans say, "a glass too many". How do you know that this girl is not some hideous monster. A black widow perhaps?

Fig 1



Bear glasses or not, no one has perfect vision. If two points in space are too close to each other, they will be perceived as a single point. In other words, if you draw two points closer and closer to each other, then it seems that there will be a distance, ϵ, such that whenever the two points are closer to each other than ϵ, they are no longer distinguishable.

For example if $\epsilon = 1$ then 0 and $\frac{3}{4}$ on the number line below are indistinguishable, whereas 0 and $1\frac{1}{2}$ are



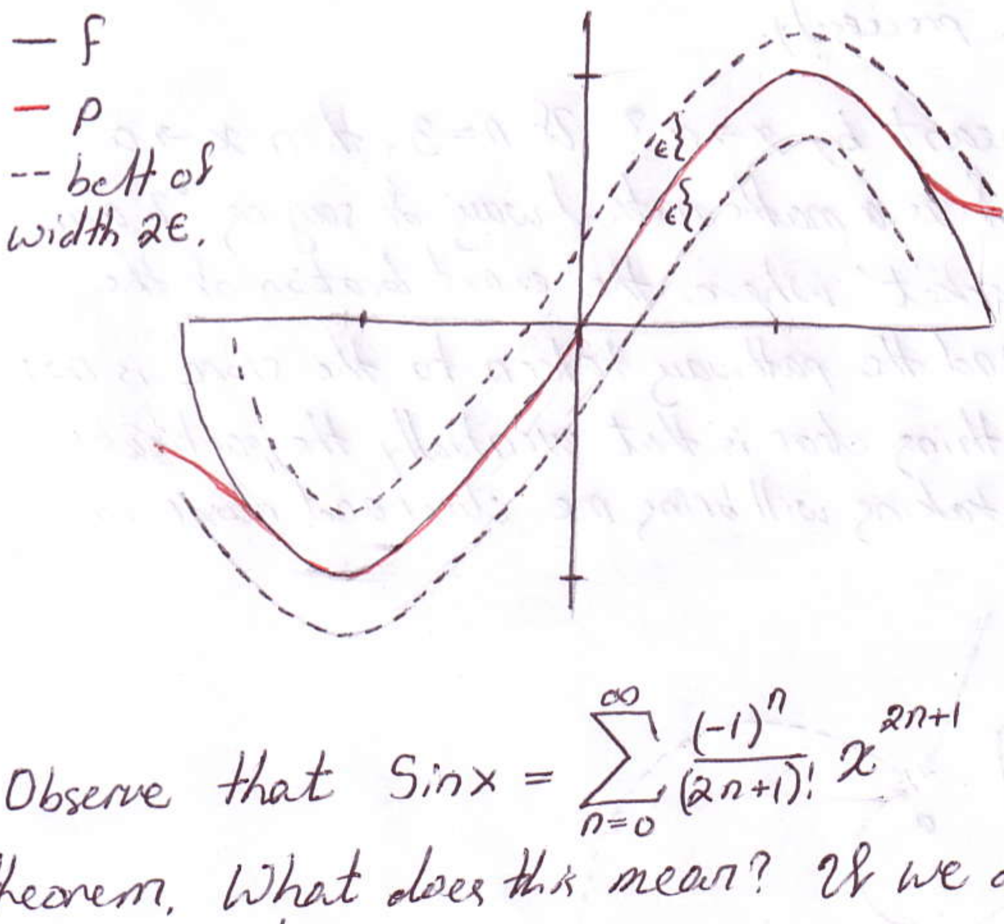
This number $\epsilon > 0$ is precicely what is meant by resolution. Notice that if ϵ is the length of the diagonal of an arbitrary square, then any two points x, y inside the square are less than a distance ϵ appart. Thus, a person with resolution ϵ will not be able to distinguish between the points x, y. To him, the girl and the spider will appear (roughly) as their respective "red squares" structures. And since these red structures are almost the same, such a man might as well marry a spider.

In analysis we are frequently concerned with measuring similarities between two functions. For example, instead of asking whether the girl is the same as the black widow spider for the purpose of marriage, we might ask if $p(x) = \sum_{n=0}^{5} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$

is similar to $f(x) = \sin x$ for $x \in [0, 2\pi]$

To answer this question, observe that $p \neq f$. However, if our eyesight is not exceptionally good, we will not see the difference. In particular, if our resolution is $\epsilon > 0$, then $p(x)$ is indistinguishable from $f(x)$ if the distance $d(f(x), p(x)) = |f(x) - p(x)| < \epsilon$. If $d(f(x), p(x)) < \epsilon$ for all $x \in [0, 2\pi]$ we may say that $d(f, p) < \epsilon$ and that f and p are indistinguishable (under resolution ϵ).

Pictorially, the 'graphs' of f and p are identical under a resolution of ϵ if they are contained within a belt of width ϵ.



Observe that $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$ by Taylor's theorem. What does this mean? If we define

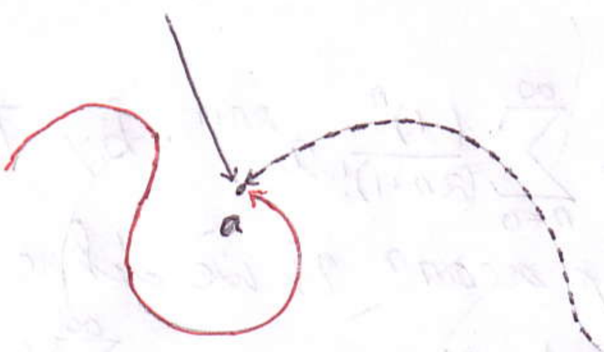
$$P_N(x) = \sum_{n=0}^{N} \frac{(-1)^n}{(2n+1)!} x^{2n+1}, \text{ then } \sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} =$$

$$= \lim_{N \to \infty} \sum_{n=0}^{N} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = \lim_{N \to \infty} P_N(x)$$

This last statement may be interpreted geometrically as one curve becomming another curve as the time frame N goes to infinity. In other words, limits allow us to understand how one object is transformed into another. Modern mathematics can describe preceizely how one function is transformed into another or how a pretty girl turns into a spider.

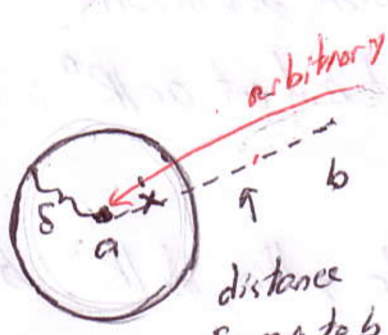
Given $f: \mathbb{R}^n \mapsto \mathbb{R}^m$, we are interested to understand statements like $\lim_{x \to a} f(x) = L$.

First what is meant by $x \to a$? If $n = 3$, then $x \to a$ might be thought of as a mathematical way of saying "I am going to the supermarket" where the exact location of the supermarket ~~is~~ a and the pathway taken to the store is not specified. The only thing clear is that eventually the particular pathway that I am taking will bring me closer and closer to position a.



Examples of pathways that I might take.

In particular, my location x will eventually be indistinguishable from location a no matter how sharp the resolution might be. That is, given $\delta > 0$, $d(x, a) < \delta$ (eventually). (1)
Another way to think of statement (1) is to observe that if my position x is to get arbitrarily close to a, then my position is to get closer to a that any stationary landmark b:



x is closer to a than δ.

Mathematically, to say that x is closer to a than any location δ units away from a is to claim that $x = (x, y, z)$ is a point in the set $\{(x, y, z) : \sqrt{(x-a_1)^2 + (y-a_2)^2 + (z-a_3)^2} < \delta\} = B_\delta(a)$
Geometrically, $B_\delta(a)$ is a ball of radius δ.

If $f : \mathbb{R}^3 \mapsto \mathbb{R}$, then we may think of the statement $\lim_{x \to a} f(x) = L$ as saying that, as the position where I stand becomes indistinguishable from a, the temperature at the location where I stand, becomes indistinguishable from temperature L. Since temperature is a scaler quantity associated with the set of real numbers $\mathbb{R}$, it may be thought of as a one dimensional space. If two temperature values are sufficiently near each other, we may not be able to distinguish between them. If ϵ represents the "temperature resolution", i.e.

If we are not able to distinguish between the temperatures T_1 and T_2 whenever $d(T_1, T_2) = |T_1 - T_2| < \epsilon$, then $\lim_{x \to a} f(x) = L$ means that, if my "temperature resolution" is ϵ, then there is a distance $\delta > 0$ s.t. if I stand closer to a than δ (i.e. my position $x \in B_\delta(a)$), the temperature at my location $f(x)$ cannot be distinguished from L (i.e $|f(x) - L| < \epsilon$).

The preceding motivates the following definitions.

Def: The open ball of radius r centered at $a \in \mathbb{R}^n$ is the set

$$B_r(a) = \{x \in \mathbb{R}^n : \|x - a\| < r\}$$

The term open ball comes from the setting of $\mathbb{R}^3$, where such objects are readily visualized.

Def: let $f : \mathcal{U} \subset \mathbb{R}^n \mapsto \mathbb{R}$

1. Let $a \in \mathbb{R}^n$ (not necessarily in $\mathcal{U}$). We say that f has limit L as x approaches a if every open ball centered at a contains points in $\mathcal{U}$ and for each positive number ϵ there corresponds a positive number δ such that $x \in B_\delta(a) \cap \mathcal{U}$ implies that $|f(x) - L| < \epsilon$.
We write $\lim_{x \to a} f(x) = L$

If there is no number L satisfying these conditions, we say $\lim_{x \to a} f(x)$ does not exist.

2. We say that $f : \mathcal{U} \subset \mathbb{R}^n \mapsto \mathbb{R}$ is continuous at the point $a \in \mathcal{U}$ if $\lim_{x \to a} f(x) = f(a)$

We say that $f : \mathcal{U} \subset \mathbb{R}^n \mapsto \mathbb{R}$ is continuous on a set S in $\mathcal{U}$

if it is continuous on every point in S. Finally, we say that f is continuous if it is continuous on its domain U.

3. We say that f has a discontinuity at a point a if it fails to satisfy one or more of the criteria for being continuous at a. We say that f has a removable discontinuity at a if $\lim_{x \to a} f(x)$ exists but f is either not defined at a or is not equal to f(a).

Ex: let $f: \mathbb{R} \mapsto \mathbb{R}$ be given by $f(x) = 3x-2$. Show that $\lim_{x \to 2} f(x) = 4$.

Solution: We must show that by standing sufficiently close to 2, f(x) will be made arbitrarily close to 4.
Suppose $|3x-2-4| < \epsilon$ then $|3x-6| = 3|x-2| < \epsilon$
Hence $|x-2| < \frac{\epsilon}{3}$. In other words, if we assume that we are already standing in such a way that $|3x-6| < \epsilon$ then we must be standing at a distance smaller than $\epsilon/3$ from 2. $\delta(\epsilon) = \epsilon/3$, is a function of ϵ that tells us how close must x be to 2 so that f(x) is closer than ϵ to 4. The existance of such a function proves that the desired limit exists.

Ex. What is $\lim_{(x,y) \to (1,0)} 2x - y$? Prove that your guess is correct.

Solution: $\lim_{(x,y) \to (1,0)} 2x - y = 2$ To prove this, we must show that for every $\epsilon > 0$ there exists δ s.t. $|2x-y-2| < \epsilon$

whenever $\|(x,y)-(1,0)\| < \delta$.

$$|2x-y-2| = |2(x-1)-y| = |(2,-1)\cdot(x-1,y)| \leq$$
$$\leq \|(2,-1)\|\|(x-1,y)\| = \sqrt{2^2+(-1)^2}\sqrt{(x-1)^2+y^2} = \sqrt{5}\sqrt{(x-1)^2+y^2} <$$
$< \epsilon$ Hence, letting $\delta(\epsilon) = \frac{\epsilon}{\sqrt{5}}$.

Comprehension check: What is $\lim\limits_{(x,y,z)\mapsto(1,-2,1)} 3x+7y-z$? Prove that your guess is correct.

The following theorem will be important at various points throughout our lectures.

Thm: If $f: U \subset \mathbb{R}^n \mapsto \mathbb{R}$ is continuous at $a \in U$ and $f(a) > 0$, then there exists $r > 0$ s.t. $f(x) > 0$ for all $x \in B_r(a) \cap U$.
(In other words, if a continuous function is positive at a point, then it must be positive on an open ball about that point)

Proof: If $f: U \subset \mathbb{R}^n \mapsto \mathbb{R}$ is continuous at a & $f(a) > 0$ then, for $\epsilon > 0$ s.t. $\epsilon = \frac{1}{2}f(a)$ $\exists\ \delta > 0$ s.t. whenever $\|x-a\| < \delta$, $|f(x)-f(a)| < \epsilon$. In particular, $\|x-a\| < \delta$ implies that $-\frac{1}{2}f(a) < f(x)-f(a) < \frac{1}{2}f(a)$ or

$$0 < \tfrac{1}{2}f(a) < f(x) < \tfrac{3}{2}f(a).$$

But $\|x-a\| < \delta$ implies that $x \in B_\delta(a)$. Thus letting $\delta = r$, the desired result follows.

<u>Remark</u>: It follows immediately from theorem above that if $f(a) < 0$, then there exists $r > 0$ s.t. $f(x) < 0$ for $x \in B_r(a)$

Thm: Suppose that $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$. Then

1. $\lim_{x\to a} (f(x)+g(x)) = L+M$

2. $\lim_{x\to a} f(x)g(x) = LM$

3. $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{L}{M}$ provided that $M \neq 0$

Proof:

1. $|f(x)+g(x)-(L+M)| = |(f(x)-L)+(g(x)-M)| \leq$

$\leq |f(x)-L| + |g(x)-M|$ (1)

where we used triangle inequality.

Because we assumed that $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$, it follows that for every $\epsilon > 0$ there exists $\delta_1, \delta_2 > 0$ s.t.

$\|x-a\| < \delta_1 \Rightarrow |f(x)-L| < \frac{\epsilon}{2}$ and $\|x-a\| < \delta_2 \Rightarrow$

$|g(x)-M| < \frac{\epsilon}{2}$. taking $\delta = \min\{\delta_1, \delta_2\}$ we see that

when $\|x-a\| < \delta$, $\|f(x)-L\| + \|g(x)-M\| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$.

This proves the desired result.

2. $|f(x)g(x) - LM| = |f(x)g(x) - Lg(x) + Lg(x) - LM|$

$\leq |f(x)g(x) - Lg(x)| + |Lg(x) - LM| = |g(x)||f(x)-L| + |L||g(x)-M|$

$= |g(x)-M+M||f(x)-L| + |L||g(x)-M| \leq$

$\leq (|g(x)-M| + |M|)|f(x)-L| + |L||g(x)-M| =$

$$= |g(x) - M||f(x) - L| + |M||f(x) - L| + |L||g(x) - M|$$

$< \epsilon^2 + |M|\epsilon + |L|\epsilon$ whenever $\|x - a\| < \delta$ for a suitable δ. Since ϵ is arbitrary, it follows that our desired result holds.

3. $$\left|\frac{f(x)}{g(x)} - \frac{L}{M}\right| = \left|\frac{Mf(x) - Lg(x)}{Mg(x)}\right| = \frac{|Mf(x) - LM + LM - Lg(x)|}{|M||g(x)|}$$

$$= \frac{1}{|M||g(x)|} |Mf(x) - LM + LM - Lg(x)| \leq$$

$$\leq \frac{1}{|M||g(x)|} \left(|M||f(x) - L| + |L||M - g(x)|\right) \quad (1)$$

Since $M \neq 0$, for $\epsilon > 0$ small enough $|g(x) - M| < \epsilon \Rightarrow$

$M - \epsilon < g(x) < M + \epsilon$ where $M + \epsilon$ and $M - \epsilon$ have the same sign (i.e. $M - \epsilon$, and $M + \epsilon$ are either both positive or both negative). In particular if $M > 0$

$$|M - \epsilon| < |g(x)| < |M + \epsilon|$$

and if $M < 0$

$$|M - \epsilon| > |g(x)| > |M + \epsilon|$$

In either case, it is posible to fix a value smaller than any $|g(x)|$ for $\|x - a\| < \delta$ where $x \in B_\delta(a) \Rightarrow |g(x) - M| < \epsilon$.

Hence $(1) \leq \frac{|M \pm \epsilon|}{|M|}\left(|M|\epsilon + |L|\epsilon\right)$

Which proves the desired result.

Recall that in Calc I you learned that polynomial functions, i.e. functions of the form $p: \mathbb{R} \mapsto \mathbb{R}$ given by $p(x) = \sum_{i=0}^{n} a_i x^i$, are continuous. Similarly, rational functions, i.e. functions of the form $r(x) = \frac{p(x)}{q(x)}$ where p, q are polynomial functions, are continuous for all x s.t. $q(x) \neq 0$. We would like to extend this result to polynomial and rational functions of many variables. To do this, we'll need the following theorem:

Thm: 1. $\lim_{x \to a} c = c$ for any constant c.

2. If $x = (x_1, x_2, \ldots, x_n)$ and $a = (a_1, a_2, \ldots, a_n)$, then $\lim_{x \to a} x_i = a_i$, $i = 1, 2, \ldots, n$. (Can you prove this?)

This theorem, along with previous results, may be used to show that a function such as $f(x_1, x_2, x_3, x_4) = x_1 x_3 + 5x_2^2 + x_4 + 1$ is continuous at any point (x_1, x_2, x_3, x_4). Furthermore, functions like $g(x_1, x_2, x_3) = \dfrac{x_1 x_2 x_3}{x_2 - 3}$ is continuous at any point (x_1, x_2, x_3) where $x_2 \neq 3$. The general result is summarized in the following theorem.

Thm: 1. If $p(x)$ is a polynomial function (i.e. one of the form

$$p(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{k} a_i x_1^{m_{1,i}} x_2^{m_{2,i}} x_3^{m_{3,i}} \cdots x_n^{m_{n,i}}, \text{ where}$$

$m_{1,i}, m_{2,i}, \ldots, m_{n,i}$, $i = 1, 2, \ldots, k$ are nonnegative integers and

$a_1, a_2, \dots, a_k$ are real numbers), then p is continuous at each point $x \in \mathbb{R}^n$.

2. If $f(x)$ is a rational function (i.e. one of the form $f(x) = \frac{p(x)}{q(x)}$, where p and q are polynomials), then f is continuous at each point in its domain. If $p(x)$ and $q(x)$ have no common factors, then these are the points where $q(x) \neq 0$. (Prove this theorem!)

Ex. The function $f: \mathbb{R}^4 \mapsto \mathbb{R}$ given by

$$f(x_1, x_2, x_3, x_4) = \frac{x_1 - 2x_2 + 3x_4}{(x_1^2 + x_2^2)(x_3 + x_4)}$$

is a rational function. This function is continuous for $x_1^2 + x_2^2 \neq 0$ and $x_3 + x_4 \neq 0$. Thus the set of points on which f is continuous is

$\{x \in \mathbb{R}^4 : x_1^2 + x_2^2 \neq 0, x_3 + x_4 \neq 0\}$

The next few examples deal with detecting removable discontinuities.

Ex. Consider $f(x,y) = \dfrac{x^4}{x^2+y^2}$. This function is continuous whenever $x^2+y^2 \neq 0$. That is, f is continuous on the set $\{(x,y) \in \mathbb{R}^2 : (x,y) \neq (0,0)\}$ $= \mathbb{R}^2 \setminus \{(0,0)\}$.

Observe that $\lim\limits_{(x,y) \mapsto (0,0)} f(x,y) = 0$;

$$0 \leq \frac{x^4}{x^2+y^2} \leq x^2 \frac{x^2}{x^2+y^2} \leq x^2 \frac{x^2+y^2}{x^2+y^2} = x^2 \text{ for all choices}$$

of (x,y) away from $(0,0)$. Since $\lim\limits_{(x,y) \mapsto (0,0)} 0 = \lim\limits_{(x,y) \mapsto (0,0)} x^2 = 0$

$\lim\limits_{(x,y) \to (0,0)} \dfrac{x^4}{x^2+y^2} = 0$ by the squeeze theorem.

Thus, the "patched-together" function

$$g(x,y) = \begin{cases} \frac{x^4}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

is continuous at $(0,0)$. In particular, the discontinuity of $f(x,y) = \frac{x^4}{x^2+y^2}$ is removable.

Certain limits are easier to compute in spherical or cylindrical coordinates rather than in Cartesian coordinate system.

Ex. $\lim\limits_{(x,y,z)\mapsto(0,0,0)} e^{\frac{-1}{x^2+y^2+z^2}} = \lim\limits_{(\rho\cos\theta\sin\varphi,\, \rho\sin\theta\sin\varphi,\, \rho\cos\varphi)\mapsto(0,0,0)} e^{\frac{-1}{\rho^2}}$

$= \lim\limits_{\rho\to 0^+} e^{-\frac{1}{\rho^2}} = 0$

Thus, by changing into spherical coordinates, the multivariate limit is reduced to a limit of one variable.

Because $\lim\limits_{(x,y,z)\mapsto(0,0,0)} e^{\frac{-1}{x^2+y^2+z^2}} = 0 \Rightarrow$ lim exists, the discontinuity of $f(x,y,z) = e^{\frac{-1}{x^2+y^2+z^2}}$ is removable

Ex. $\lim\limits_{(x,y)\mapsto(0,0)} \frac{\sin(x^2+y^2)}{x^2+y^2} = \lim\limits_{(r\cos\theta,\, r\sin\theta)\mapsto(0,0)} \frac{\sin((r\cos\theta)^2+(r\sin\theta)^2)}{(r\cos\theta)^2+(r\sin\theta)^2} =$

$= \lim\limits_{r\mapsto 0^+} \frac{\sin(r^2)}{r^2} = 1$

Here, we used polar coordinates to reduce the multivariate limit to the familiar limit of Calc I.

More generally, if $\lim\limits_{(x_1,x_2,\dots,x_n)\mapsto(a_1,a_2,\dots,a_n)} f(x_1,x_2,\dots,x_n)$ can be expressed as $\lim\limits_{(x_1,x_2,\dots,x_n)\mapsto(a_1,a_2,\dots,a_n)} g(p(x_1,x_2,\dots,x_n))$ where

$\lim\limits_{(x_1,\dots,x_n)\mapsto(a_1,a_2,\dots,a_n)} p(x_1,x_2,\dots,x_n) = A$ is known and $g: \mathbb{R}\mapsto\mathbb{R}$ is a function of one variable

then $\lim\limits_{(x_1,x_2,\dots,x_n)\mapsto(a_1,a_2,\dots,a_n)} f(x_1,x_2,\dots,x_n) = \lim\limits_{u\mapsto A} g(u)$.

Ex. Compute $\lim\limits_{(x,y,z)\mapsto(1,1,-6)} (6x^3y+z)\,\mathrm{Sin}\left(\frac{z}{6x^3yz+z^2}\right)$

Solution: After simplifying the rational expression inside Sin, we obtain

$$\lim_{(x,y,z)\mapsto(1,1,-6)} (6x^3y+z)\,\mathrm{Sin}\left(\frac{1}{6x^3y+z}\right)$$

let $g(u) = u\,\mathrm{Sin}\left(\frac{1}{u}\right)$ and let $p(x,y,z) = 6x^3y+z$.

then p is a multivariate polynomial. Since polynomials are continuous, $\lim\limits_{(x,y,z)\mapsto(1,1,-6)} 6x^3y+z = 6(1)^3(1)+(-6) = 0$

Hence, $\lim\limits_{(x,y,z)\mapsto(1,1,-6)} (6x^3y+z)\sin\left(\frac{1}{6x^3y+z}\right) =$

$= \lim\limits_{u\to 0} u\sin\left(\frac{1}{u}\right) = 0$ by calc. I methods.

Comprehension Check: Reduce $\lim\limits_{(x,y)\mapsto(0,0)} \frac{\sin(6xy)}{3xy}$ to a single-variate limit and solve. What would you do if you needed to compute $\lim\limits_{(x,y)\mapsto(0,0)} \frac{\sin(xy)}{y}$?

Ex. the function $f(x,y) = \frac{x^2-y^2}{x^2+y^2}$ has a discontinuity at $(0,0)$. To see that this discontinuity is not removable, we will show that $\lim\limits_{(x,y)\mapsto(0,0)} \frac{x^2-y^2}{x^2+y^2}$ does not exist.

Consider what happens if (x,y) approaches $(0,0)$ along the x-axis. At such points, $y=0$. We see that

$$\lim_{\substack{(x,y)\mapsto(0,0)\\ y=0}} \frac{x^2-y^2}{x^2+y^2} = \lim_{x\to 0} \frac{x^2-0}{x^2+0} = 1$$

On the other hand, approaching along the y-axis we see that

$$\lim_{\substack{(x,y)\mapsto(0,0)\\ x=0}} \frac{x^2-y^2}{x^2+y^2} = \lim_{y\mapsto 0} \frac{0-y^2}{0+y^2} = -1$$

Since this does not agree with 1, the given limit does not exist.

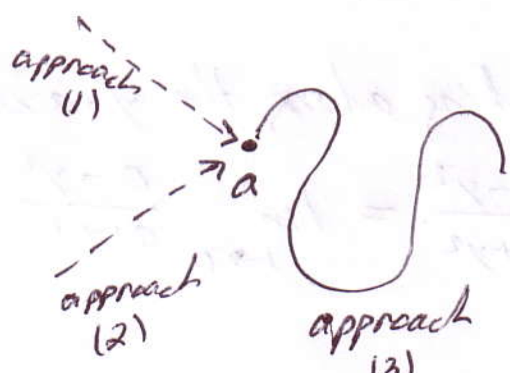
This could have also been detected by trying to compute the given limit in another coordinate system:

$$\lim_{(x,y)\mapsto(0,0)} \frac{x^2-y^2}{x^2+y^2} = \lim_{(r\cos\theta,\, r\sin\theta)\mapsto(0,0)} \frac{(r\cos\theta)^2-(r\sin\theta)^2}{(r\cos\theta)^2+(r\sin\theta)^2} =$$

$$= \lim_{r\mapsto 0} \frac{r^2(\cos^2\theta - \sin^2\theta)}{r^2} = \lim_{r\mapsto 0} (\cos^2\theta - \sin^2\theta) =$$

$= \lim_{r\mapsto 0} \cos 2\theta$. This means that it matters under which angle the origin is approached. For example, if we approach at an angle of $\frac{\pi}{6}$ (i.e. along a line $y = (\tan\frac{\pi}{6})x$ $\Rightarrow y = \frac{1}{\sqrt{3}}x$) our limit will equal to $\cos(2\frac{\pi}{6}) = \cos(\frac{\pi}{3}) = \frac{1}{2}$.

In single variable calculus $\lim_{x\to a} g(x) = L$ iff $\lim_{x\to a^+} g(x) = \lim_{x\to a^-} g(x) = L$. because in a 1-D space there is only one type of motion, namely motion along a line. The point a can therefore be approached from above or from below. In 2-D space, the situation is already much more difficult.



Not only can a be approached along any line through a, but also along any conceivable curve through a. That makes for a hell-of-a-lot "directions" to check to be sure that the limit exists.

Can't we just check the approaches along lines? The following example provides an answer.

Ex. Consider $\lim\limits_{(x,y)\mapsto(0,0)} \frac{x^2y}{x^4+y^2}$. We can see what happens when we approach $(0,0)$ along a generic line $y=mx$

$$\lim_{\substack{(x,y)\mapsto(0,0)\\ y=mx}} \frac{x^2y}{x^4+y^2} = \lim_{x\mapsto 0} \frac{x^2mx}{x^4+(mx)^2} = \lim_{x\mapsto 0} \frac{mx^3}{x^4+mx^2} =$$

$$= \lim_{x\to 0} \frac{mx}{x^2+m} = 0$$ (You should also check that

$$\lim_{\substack{(x,y)\mapsto(0,0)\\ y=0}} \frac{x^2y}{x^4+y^2} = \lim_{\substack{(x,y)\mapsto(0,0)\\ x=0}} \frac{x^2y}{x^4+y^2} = 0)$$

Thus, approaching $(0,0)$ along any line leads to the same limit.

Does this mean that $\lim\limits_{(x,y)\mapsto(0,0)} \frac{x^2y}{x^4+y^2} = 0$?

Consider $\lim\limits_{\substack{(x,y)\mapsto(0,0)\\ y=x^2}} \frac{x^2y}{x^4+y^2} = \lim\limits_{x\to 0} \frac{x^2x^2}{x^4+(x^2)^2} = \lim\limits_{x\to 0} \frac{x^4}{2x^4} = \frac{1}{2} \neq 0$

In particular, IT MATTERS if we approach along lines or curves.

Comprehension check:

Identify two curves $y=g(x)$ and $y=p(x)$ s.t.

$$\lim_{\substack{(x,y)\to(0,0)\\ y=g(x)}} \frac{x^9y^3}{2x^{18}+y^6} \neq \lim_{\substack{(x,y)\mapsto(0,0)\\ y=p(x)}} \frac{x^9y^3}{2x^{18}+y^6}$$

Limits of Vector-Valued Functions

In many applications, it is important to consider limits of functions of the form $f: U\subset\mathbb{R}^n \mapsto \mathbb{R}^m$.

For example, suppose U is the set of all points on a piece of rail together with a measurement of temperature at each point.

Let $f_T: U\subset\mathbb{R}^4 \mapsto \mathbb{R}^3$ be a function that determines what happens to each point on the rail if the temperature is increased by T degrees. In other words, if $U_c = \{(x,y,z) : (x,y,z) \text{ is a point on the rail when temperature } t=c\}$ then U_c can be thought of as $\{(x,y,z,t): t=c\}\cap U$ and $f(U_c) \equiv$

$\equiv$ transformation in the shape of the rail.

$\mathcal{U}_c \equiv$ 

rail's shape when $t=c$

$f_T(\mathcal{U}_c) \equiv$ 

rail's shape when $t=c+T$

then $\lim\limits_{(x,y,z,t)\mapsto(a,b,c,d)} f(x,y,z,t)$ represents the "distortion" of points with initial temperature near d and position (x,y,z) near (a,b,c).

Def: Let $f: \mathcal{U} \subset \mathbb{R}^n \mapsto \mathbb{R}^m$ and let $a \in \mathbb{R}^n$

1. We say $\lim\limits_{x\to a} f(x) = L$ if $B_\delta(a) \cap \mathcal{U}$ is nonempty for all $\delta > 0$ and for each positive number ϵ there exists a positive number δ such that $\|f(x) - L\| < \epsilon$ whenever $x \in B_\delta(a) \cap \mathcal{U}$
2. We say that f is continuous at a if $a \in \mathcal{U}$ and $\lim\limits_{x\to a} f(x) = f(a)$

Any function $f: \mathcal{U} \subset \mathbb{R}^n \to \mathbb{R}^m$ has m real-valued component functions: $f(x) = (f_1(x), f_2(x), \dots, f_m(x))$. Does the continuity of f has any relation to the continuity of the component functions? The answer is provided in the following theorem:

Thm: If $f: \mathcal{U} \subset \mathbb{R}^n \to \mathbb{R}^m$ is given by

$$f(x) = (f_1(x), f_2(x), \dots, f_m(x))$$

then f is continuous at a point $a \in \mathcal{U}$ iff $f_1, f_2, \dots, f_m$ are all continuous at a.

Proof: Suppose f is continuous at $a \in \mathbb{R}^n$. Then there exists a δ s.t. $x \in B_\delta(a) \Rightarrow \|f(x) - f(a)\| < \epsilon$.

Hence $$|f_i(x) - f_i(a)| \leq \sum_{i=1}^{m} |f_i(x) - f_i(a)|^2 = \|f(x) - f(a)\| < \epsilon$$

for any $i \in \{1, 2, \dots, m\}$. This shows that f_i is continuous at a.

Suppose now that each of the component functions f_i are continuous at $a \in \mathbb{R}^n$. Then for $\epsilon > 0$, there exists $\delta_i > 0$ s.t. $|f_i(x) - f_i(a)| < \frac{\epsilon}{m}$ whenever $x \in B_{\delta_i}(a)$. Let $\delta = \min\{\delta_1, \delta_2, \dots, \delta_m\}$, then $x \in B_\delta(a)$ implies that $|f_i(x) - f_i(a)| < \frac{\epsilon}{m}$ for all i.

By the triangle inequality

$$\|f(x) - f(a)\| \leq \sum_{i=1}^{m} |f_i(x) - f_i(a)| < \sum_{i=1}^{m} \frac{\epsilon}{m} = m \frac{\epsilon}{m} = \epsilon.$$

This proves that f is continuous.

Ex. $f(x,y) = \left(\frac{xy-y}{xy-x}, \frac{1}{x^2+y^2-9} \right)$

then f is continuous precisely at those points $\vec{a} = (a_1, a_2)$ for which $f_1(x,y) = \frac{xy-y}{xy-x}$ and $f_2(x,y) = \frac{1}{x^2+y^2-9}$ are continuous.

f_1 is continuous when $xy - x \neq 0 \Rightarrow x(y-1) \neq 0$ or when $x \neq 0$ and $y \neq 1$.

f_2 is continuous when $x^2+y^2-9 \neq 0$ or when $x^2+y^2 \neq 9$.

Hence f is discontinuous on the x and y axis and on the circle centered at the origin with radius $r=3$.

The composition of two continuous functions is continuous, as the following theorem proves.

Thm: Let $f: U \subset \mathbb{R}^n \to V \subset \mathbb{R}^m$ and $g: V \subset \mathbb{R}^m \to \mathbb{R}^p$. Suppose f is continuous at $a \in U$ and g is continuous at $f(a)$. then the function $g \circ f$ defined by $(g \circ f)(x) = g(f(x))$ is continuous at a.

Proof: We must show that $\lim_{x \to a} g(f(x)) = g(f(a))$.

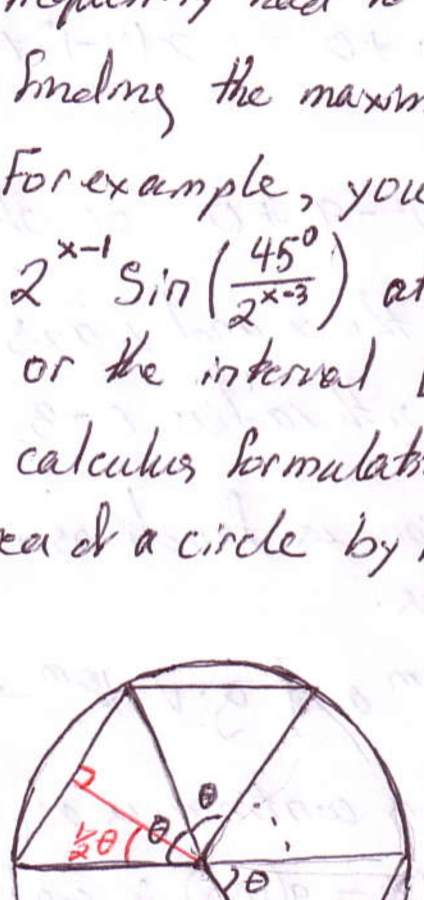
To do this, observe that $\|f(x) - f(a)\| < \delta_f$ whenever δ is sufficiently small and $x \in B_\delta(a) \cap U$. However if δ_f is small $\|g(y) - g(f(a))\| < \epsilon$ for all $y \in B_{\delta_f}(f(a)) \cap V$.

Hence $\|g(f(x)) - g(f(a))\| < \epsilon$ for $x \in B_\delta(a) \cap U$.

Ex. The function $\sin(3xy^2-7x^3) = h(x,y)$ is continuous, because it is the composition of $p(x,y) = 3xy^2 - 7x^3$ and $q(x) = \sin x$ both of which are continuous.

Open sets, closed sets, and continuity

In Calc I you frequently had to consider optimization problems like finding the maximum and minimum values of a function. For example, you might ask if the function $f(x) = 2^{x-1}\sin\left(\frac{45^0}{2^{x-3}}\right)$ attains a maximum on the interval $(0,\infty)$ or the interval $[3,4]$. This question may be viewed as a calculus formulation of the geometric problem of finding the area of a circle by means of polygones.



Circle of radius r approximated by polygon with $\frac{360}{\theta}$ sides.

If we approximate the circle by a polygon with $\frac{360}{\theta}$ sides of equal length, then the area of one triangular segment of the polygon is $r\sin(\frac{\theta}{2})\, r\cos(\theta/2) = \frac{r^2}{2}\sin\theta$. The area of the polygon is consequently $\frac{360}{\theta}\frac{r^2}{2}\sin\theta$. If we consider polygons with 2^n sides of equal length, their area is given by the formula

$$2^n \frac{r^2}{2}\sin\left(\frac{360^\circ}{2^n}\right) = \left(2^{n-1}\sin\left(\frac{45^\circ}{2^{n-3}}\right)\right) r^2 = f(n)\, r^2.$$

Because the interval $[3,4]$ is closed, $f(x)$ attains both max and min over this interval. Furthermore, geometric considerations will establish that f is increasing so the maximum will occur at $x=4$ and the minimum will be at $x=3$.

Although f is continuous, it does not attain a maximum over the interval $(0,\infty)$, because $f(n+1) > f(n)$ for any integer $n \geq 3$. What is the difference between $[3,4]$ and $(0,\infty)$? One interval is bounded, the other isn't. One interval is closed and the other is opened.

Generally, to describe the behavior of a continuous function, we will need to know whether its domain is a closed or open set and whether it is bounded or not. Before we extend the notion of open and closed intervals to open and closed subsets of $\mathbb{R}^n$, let us be convinced that this journey is really worth the effort.

Can the area of a circle be really approximated by polygons?

$n=2$ $\qquad$ $n=3$

In the figure above, the circle on the left is approximated by $f(2) r^2$ and the one on the right is approximated by $f(3) r^2$. Notice that the walls of the polygon on the right are closer to the walls of the circle in comparison to the picture on the left. Let ε_n denote the shaded region between one wall of a 2^n sided polygon and the circle. The figure above suggests that $\varepsilon_2 > \varepsilon_3 > \dots > \varepsilon_n > \dots$ However a polygon with 2^{n+1} sides will have twice as many shaded regions as a polygon with 2^n sides. In particular, $2^n \varepsilon_n$ = Area of circle − Area of polygon with 2^n sides.

How can we be sure that $\lim_{n \to \infty} 2^n \varepsilon_n = 0$? Geometrically this is the same as asking which curves are rectifiable (i.e. which curves can be approximated to any degree of accuracy by line segments).

A fundamental theorem in analysis, called the Heine-Borel theorem, identifies a surface as rectifiable, whenever it is the image of a closed and bounded set under a continuous map.

The upper part of the circle of radius r can be described by the vector valued function $g(x) = (x, \sqrt{r^2 - x^2})$ $x \in [-r, r]$.

Since $[-r, r]$ is closed and bounded, we can be sure that

Area of circle of radius $r = C(r) = \lim_{n \to \infty} f(n) r^2$.

Indeed, important theorems like the fundamental theorem of calculus and the mean-value theorem itself depend on the consequences of the Heine-Borel theorem.

What is π? (optional)

It seems that $\pi = \lim_{n\to\infty} f(n)$ would be a good definition since $C(r) = \left(\lim_{n\to\infty} f(n)\right) r^2$.

Observe that $\sin\left(\frac{45^\circ}{2^{n-3}}\right) = \sin\left(\frac{45^\circ}{2\,2^{n-4}}\right) = \sqrt{\frac{1-\cos\left(\frac{45^\circ}{2^{n-4}}\right)}{2}}$.

In particular, $\sin\left(\frac{45^\circ}{2}\right) = \sqrt{\frac{1-\cos(45^\circ)}{2}} = \sqrt{\frac{2-\sqrt{2}}{4}} = \frac{\sqrt{2-\sqrt{2}}}{2}$

$$\sin\left(\frac{45^\circ}{4}\right) = \sqrt{\frac{1-\cos\left(\frac{45^\circ}{2}\right)}{2}} = \sqrt{\frac{1-\sqrt{1-\frac{2-\sqrt{2}}{4}}}{2}} =$$

$$= \sqrt{\frac{1-\sqrt{\frac{2+\sqrt{2}}{4}}}{2}} = \sqrt{\frac{2-\sqrt{2+\sqrt{2}}}{4}} = \frac{\sqrt{2-\sqrt{2+\sqrt{2}}}}{2}$$

Hence $\pi \approx 2^{n-2}\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2+\cdots}}}}$ whenever n is large.

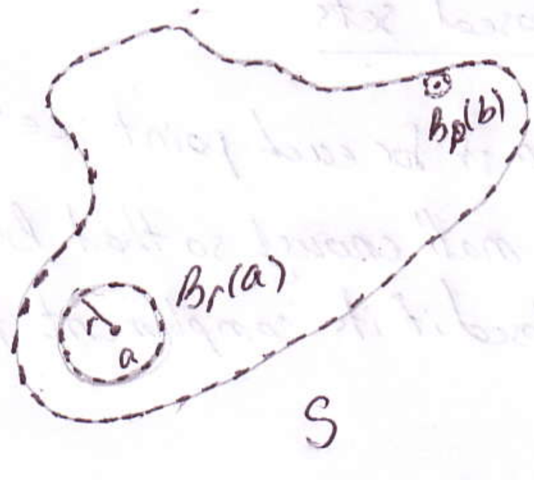
Open & closed sets

Def: A set $S \subset \mathbb{R}^n$ is open if for each point $x \in S$ there is a positive number δ that is mall enough so that $B_\delta(x)$ is contained in S. A set S in $\mathbb{R}^n$ is closed if its complement $\mathbb{R}^n \setminus S =$
$= \{x \in \mathbb{R}^n;\ x \notin S\}$ is open.

Proposition: Let $a \in \mathbb{R}^n$, then the open ball $B_r(a)$ is an open set.

Proof: Suppose $x \in B_r(a)$, we must show that there is $\delta > 0$ such that $B_\delta(x) \subset B_r(a)$. Since $x \in B_r(a)$, we know that $\|x-a\| < r$ let $\delta = r - \|x-a\|$. Then $\delta > 0$ and consider $B_\delta(x)$. To show that it is a subset of $B_r(a)$, pick $y \in B_\delta(x)$. Then $\|y-a\| = \|y-x+x-a\| = \|(y-x)+(x-a)\| \leq \|y-x\| + \|x-a\| < \delta + \|x-a\| = \delta + r - \delta = r$. Where we used triangle inequality. Hence $y \in B_r(a) \Rightarrow B_\delta(x) \subset B_r(a)$.

Intuitively speaking, open sets are sets that have none of their points in contact with the outside. In other words, any point of an open set S is an interior point.
You can think of them as amebas without a well-defined membrane:

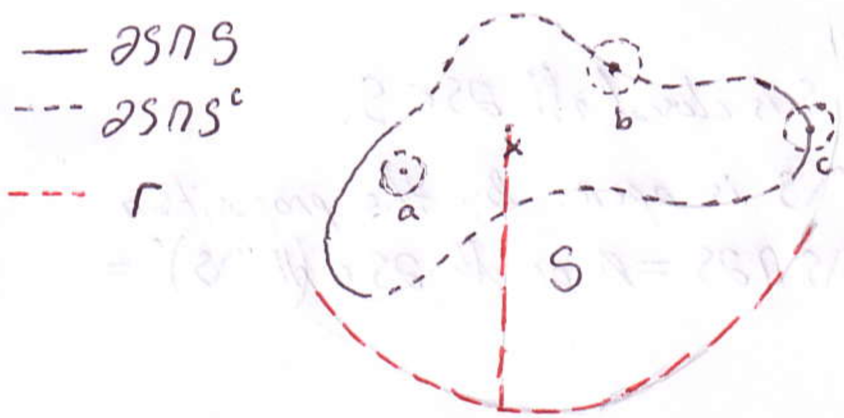


This "membrane" is called the boundary of S and what it encloses is the interior of the set.

Def: The boundary of a set S in $\mathbb{R}^n$ is the set $\partial S =$
$= \{x \in \mathbb{R}^n ;\ B_\delta(x) \cap S \text{ and } B_\delta(x) \cap S^c \neq \emptyset\}$. In other words
$x \in \partial S$ if x is arbitrarily close to points in S and points outside of S (i.e. in $S^c = \mathbb{R}^n \setminus S$).

The interior of a set S, denoted $int(S)$, is the portion of the set remaining after its boundary has been removed.

A set $S \subset \mathbb{R}^n$ is bounded if for any $x \in S$ there exists $r > 0$ such that $S \subset B_r(x)$.



Notice that in the figure above $a \in int(S)$, $b \in \partial S \cap S^c$ and $c \in \partial S \cap S$. That is a is an interior point, b is a point on the boundary of S that is not part of S, and c is a point of S that lies on the boundary.

Observe that S is not open, because $B_\delta(c) \not\subset S$ no matter how small δ might be. In particular, for a set to be open, its intersection with the boundary must be the empty set.

Proposition: let $S \subset \mathbb{R}^n$. Then S is open iff $S \cap \partial S = \emptyset$

Proof: Suppose S is open, then for any $x \in S$ there is a $\delta > 0$ s.t. $B_\delta(x) \subset S$. But this means that $B_\delta(x) \cap \mathbb{R}^n \setminus S = \emptyset$, which implies that $x \notin \partial S$. In particular $S \cap \partial S = \emptyset$.

Suppose now that $S \cap \partial S = \emptyset$. Then for any point $x \in S$, it is not the case that $B_\delta(x) \cap S^c \neq \emptyset$ for every δ. Hence there must be some $\delta > 0$ such that $B_\delta(x) \cap S^c = \emptyset$ which imediately implies that $B_\delta(x) \subset S$. Hence, in that case, S is open.

This completes the proof.

Corrolary: Let $S \subset \mathbb{R}^n$. Then S is closed iff $\partial S \subset S$.

Proof: S is closed iff $S^c = \mathbb{R}^n \setminus S$ is open. By the proposition above, $\mathbb{R}^n \setminus S$ is open iff $\mathbb{R}^n \setminus S \cap \partial S = \emptyset$ or iff $\partial S \subset (\mathbb{R}^n \setminus S)^c =$
$= \mathbb{R}^n \setminus (\mathbb{R}^n \setminus S) = S$.

How can we decide whether a given set is open, closed or neither? The following proposition is rather useful and it can be easily proved. (you should supply the proof).

Proposition: let $S_1, S_2, \dots, S_p$ be open sets of $\mathbb{R}^n$ and let $C_1, C_2, \dots, C_q$ be closed subsets of $\mathbb{R}^n$. Then

1. $\bigcup_{i=1}^{p} S_i$ is open
2. $\bigcap_{i=1}^{p} S_i$ is open ($\emptyset$ is both open and closed by def)
3. $\bigcap_{i=1}^{q} C_i$ is closed
4. $\bigcup_{i=1}^{q} C_i$ is closed

Ex. Let $C = \{(x,y) \in \mathbb{R}^2 ; x^2+y^2 = 1\}$. Show that C is closed.

Solution: We can show that C is closed by showing that $\mathbb{R}^2 \setminus C$ is open. Observe that $\mathbb{R}^2 \setminus C = A \cup B$ where $A = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 < 1\}$ and $B = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 > 1\}$

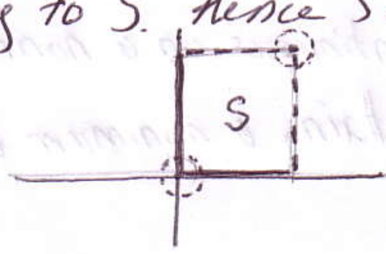
Since $A = B_1(0,0)$, we know that A is open. To see that B is open pick any $(x,y) \in B$. then $\|(x,y)\| = \sqrt{x^2+y^2} > 1$. Hence $\|(x,y)\| = 1+\delta$, where $\delta > 0$ we will show that $B_\delta(x,y) \subset B$.

To see this, pick $(a,b) \in B_\delta(x,y)$. Then $\|(a,b)\| = \sqrt{a^2+b^2} > 1$. because, if $\|(a,b)\| \leq 1$ then $1+\delta = \|(x,y)\| \leq \|(a,b)-(x,y)\| + \|(a,b)\| <$
$< \delta + \|(a,b)\| \leq \delta + 1$ which is a contradiction (since $1+\delta$ cannot be less than itself), Hence $B_\delta(x,y) \subset B$ and B is open as desired.

Since $\mathbb{R}^2 \setminus C = A \cup B$ which is a union of open sets, it is an open set, Therefore C is closed.

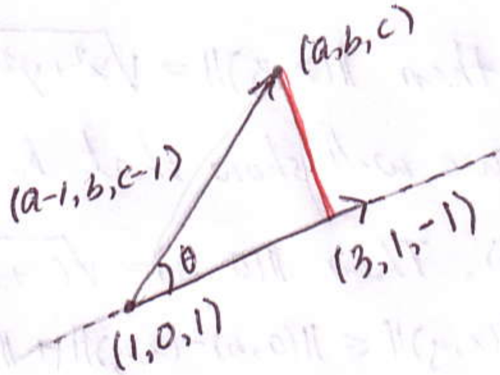
<u>Remark</u>: Sets are not doors! They can be open, closed, or neither.

Ex. The set $S = \{(x,y) ; 0 \leq x < 1, 0 \leq y < 1\}$ is neither open nor closed. To see that S is not opened observe that $(0,0) \in S \cap \partial S \neq \emptyset$. To see that S is not closed observe that $(1,1) \in \partial S$ but does not belong to S. Hence S does not contain its boundary.



Ex. Show that the line $L(t) = (3t+1, t, -t+1)$ is a closed subset of $\mathbb{R}^3$.

Solution: The line is the set $L = \{(x,y,z) : (x,y,z) = (3t+1, t, -t+1), t \in (-\infty, \infty)\}$. Let's show that L^c is open. pick $(a,b,c) \in L^c$, Then the distance from (a,b,c) to the line L is determined from the diagram below:



The distance is $\|(a-1,b,c-1)\| \sin\theta$ where θ is the angle between $(a-1,b,c-1)$ and $(3,1,-1)$. Hence the distance is $\frac{\|(a-1,b,c-1)\| \|(3,1,-1)\| \sin\theta}{\|(3,1,-1)\|}$

$$= \frac{\|(a-1,b,c-1) \times (3,1,-1)\|}{\|(3,1,-1)\|} = \delta.$$ Observe that $B_\delta(a,b,c) \cap L = \emptyset$

Hence $B_\delta(a,b,c) \subset L^c$, which implies that L^c is open, which in turn implies that L is closed.

The main thing that we need to know in this class is summarized in this theorem:

Thm: If $f : U \subset \mathbb{R}^n \to \mathbb{R}$ is continuous on a nonempty closed and bounded subset S of U, then f attains a minimum and a maximum value on S.