(2.1)

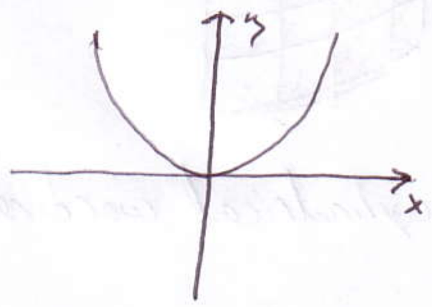
Graphs of functions and sets

Draw the graph of $y=x^2$

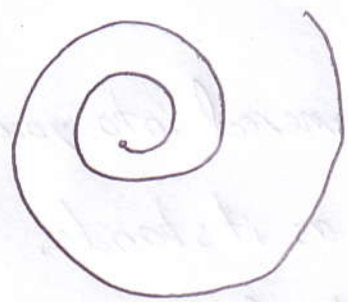
Contrary to what was hammered into your heads in school, this request is unreasonable as it stands.

First, you should realize that, if we are in $\mathbb{R}^2$, $y=x^2$ stands for the set of all pairs (x,y) that satisfy $y=x^2$. That is $P=\{(x,y): y=x^2\} = \{(x,x^2): x\in\mathbb{R}\}$. The property $y=x^2$ is a test criterion for the inclusion or exclusion of (x,y) from P. For instance, $(2,4)\in P$ because the second coordinate 4 is equal to the square of the first coordinate. Conversely, $(-7,46)\notin P$. (why?).

This still doesn't settle the problem of drawing a graph. The set P is a collection of coordinates, to translate these coordinates into drawing, we must give them physical interpretation. If P is thought to be a collection of rectangular coordinates, then the drawing is a familiar parabola



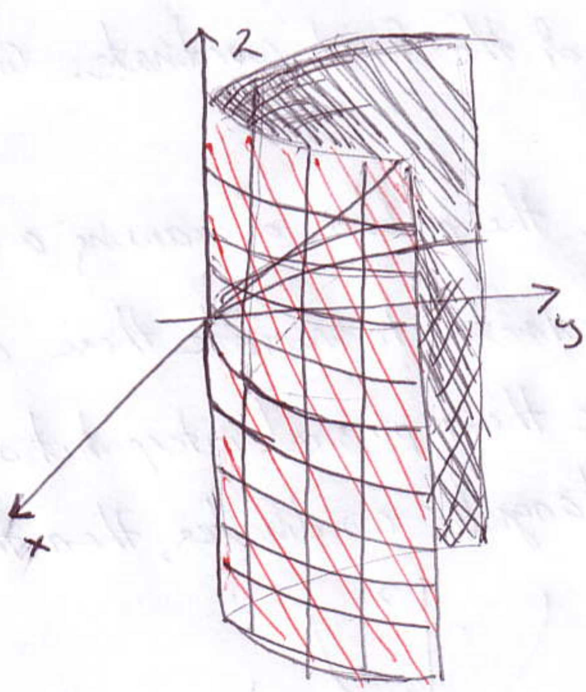
If we think of P as a collection of polar coordinates, we get a completely different drawing.



If we think of $y=x^2$ as a property of some subset of $\mathbb{R}^3$

Then $P = \{(x,y,z) : y = x^2\} = \{(x, x^2, z) : x, z \in \mathbb{R}\}$

In rectangular coordinates, the drawing becomes



What is the drawing in cylindrical coordinates?

Unless otherwise specified, when a drawing of the graph is required, we will assume that the underlying coordinate system is rectangular.

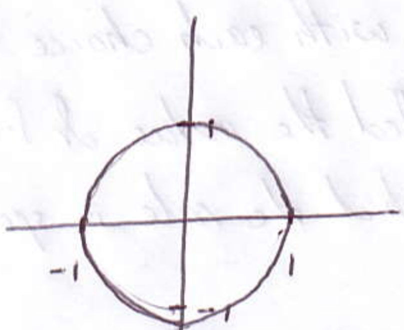
Ex. Draw $x^2+y^2=1$ in

(a) $\mathbb{R}^2$ and

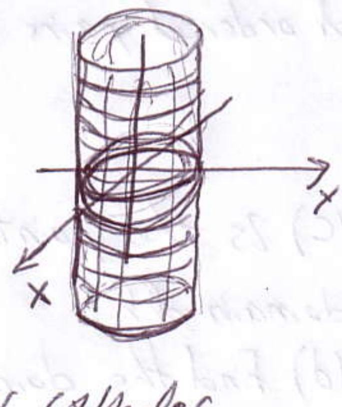
(b) $\mathbb{R}^3$

Solution:

(a) this is simply the unit circle, or a circle of radius $r=1$ and center at $(0,0)$



(b) The set of points in $\mathbb{R}^3$ that satisfy $x^2+y^2=1$ is of the form $\{(x,y,z) : x^2+y^2=1\}$. In other words, z is free to assume any value.



The drawing is a circular cylinder.

Comprehension check: Draw in rectangular coordinates.

(a) $x^2+z^2=1$

(b) $z=y+1$.

Please note, graphs and drawings of graphs are not the same thing. One graph might correspond to no drawings or many (very distinct) drawings.

Functions of two variables

Def: A scalar-valued function of two real variables x and y is a rule f, that associates with each choice of x and y a single real number $f(x,y)$ called the value of f at (x,y). The set of ordered pairs (x,y) for which the rule is specified is the domain of the function.

Ex. $f(x,y)=x^2+y^2$ associates with x and y the sum of the squares of x and y.

$f(2,-5) = 2^2+(-5)^2=29$

The domain of f is the set of ordered pairs $(x,y)\in\mathbb{R}^2$ (why?)

Ex. Let $f(x,y)=\dfrac{\ln y}{x}$.

(a) What is $f(-3,2)$?

(b) What is $f(4,e)$?

(c) Is the point $(e^{-3},-2)$ in the domain of f?

(d) Find the domain of f.

Solution:

(a) $f(-3,2) = \frac{\ln 2}{-3} = -\frac{\ln 2}{3}$

(b) $f(4,e) = \frac{\ln e}{4} = \frac{1}{4}$

(c) $(e^{-3},-2)$ is not in the domain of f, because $\ln(-2)$ is not defined.

(d) For $f(x,y) = \frac{\ln y}{x}$ to be defined, $x \neq 0$ and $y > 0$

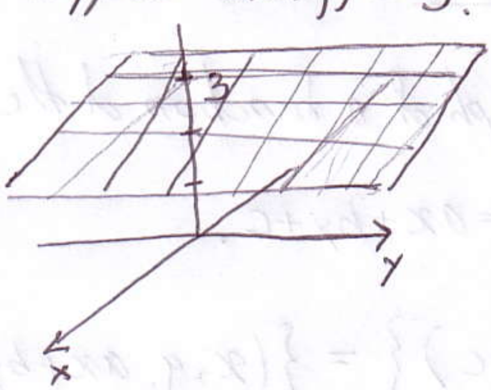
Hence, the domain is $\{(x,y) : x \neq 0,\ y > 0\}$

Graphs of Functions of two Variables

Def: The graph of a function f of two variables is the set of all points (x,y,z) in $\mathbb{R}^3$ such that $z = f(x,y)$ and (x,y) is in the domain of f.

Ex. Let c be any constant, $c \in \mathbb{R}$, the function $f(x,y) = c$ has $\mathbb{R}^2$ as its domain. Its graph is $\{(x,y,z) : c = z\} = \{(x,y,c)\}$.

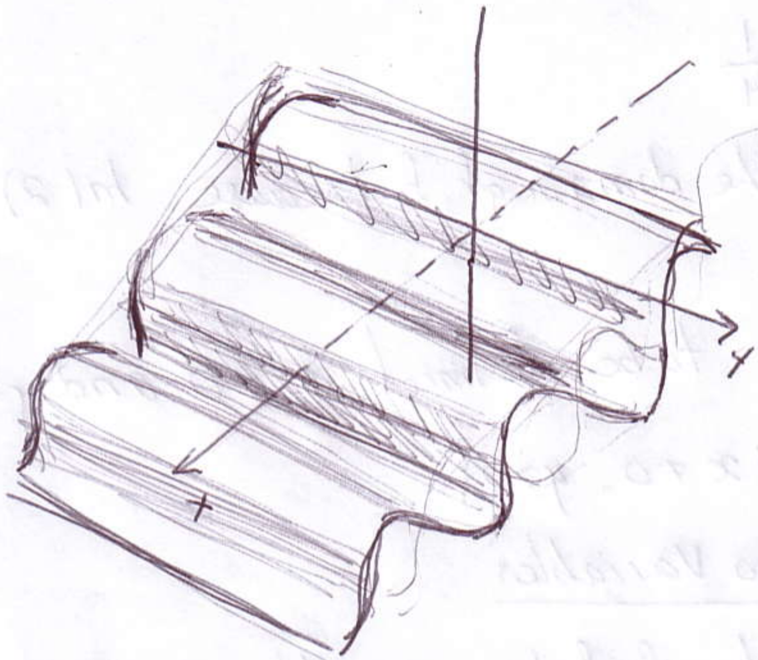
Comprehension check: Suppose $f(x,y) = 3$. Is



the graph of f? Why or why not.

Ex. Draw the graph of $f(x,y) = \sin x$.

Solution: z depends only on x. No matter what y is, $f(x,y) = \sin x$



In general, if f is a function of two variables in which one of the variables does not appear, then the drawing of the graph is the union of parallel copies of a single curve that lies in either the xz-, yz, or xy-plane. The graph of a function of two variables in which one of the variables does not appear is called a cylinder.

Planes

The picture of the graph of a function of the form (in rectangular coordinates) $\quad p(x,y) = ax+by+c$.

$$\{(x,y,z=ax+by+c)\} = \{(x,y,ax+by+c)\} =$$
$$= \{(x,0,ax)+(0,\cancel{by},by)+(0,0,c)\} = \{x(1,0,a)+y(0,1,b)+(0,0,c)\}$$

In other words, the graph of $p(x,y)$ is the plane through the point $(0,0,c)$ spanned by the vectors $(1,0,a)$ and $(0,1,b)$

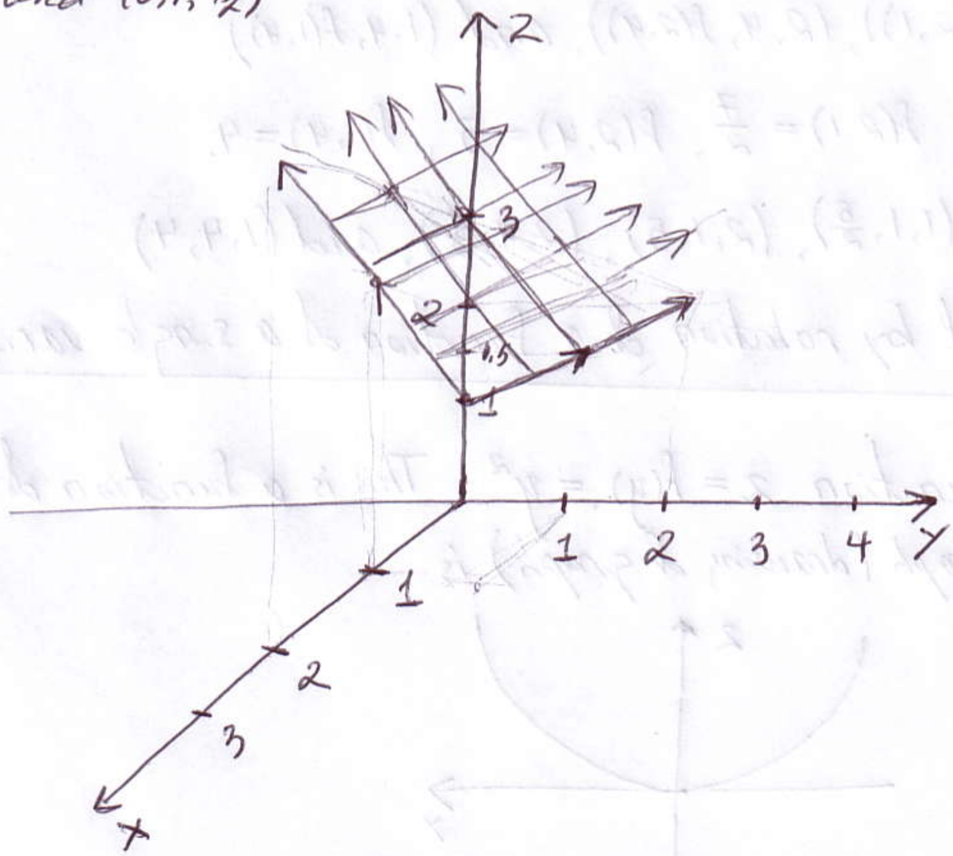
Ex. (a) Draw the plane $f(x,y) = x + \frac{1}{2}y + 1$

(b) Draw the portion of f that lies above the rectangle $R = \{(x,y) : 1 \le x \le 2,\ 1 \le y \le 4\}$

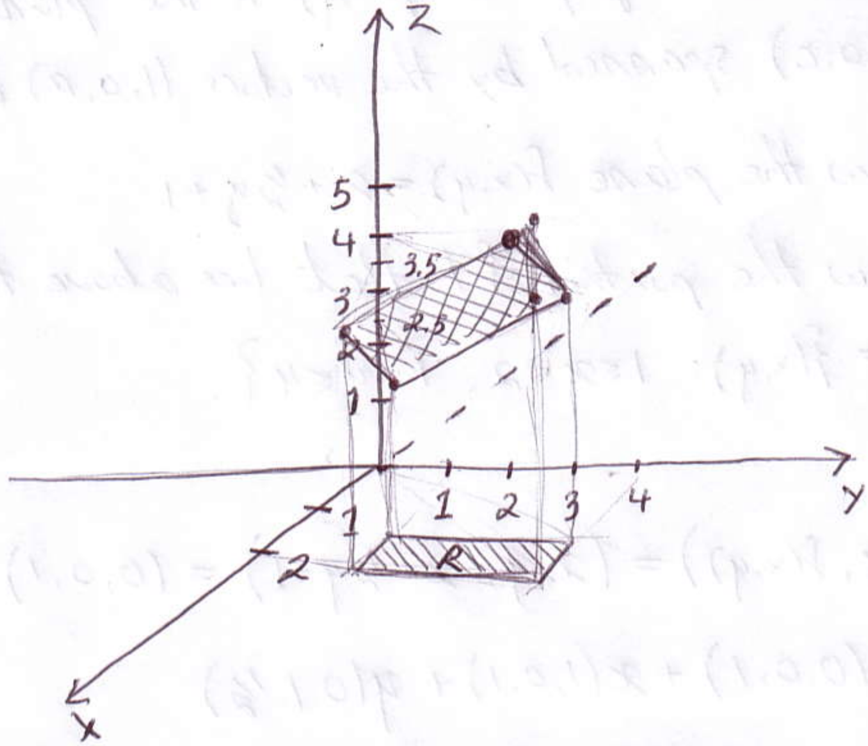
Solution:

(a) $(x, y, f(x,y)) = (x, y, x + \frac{1}{2}y + 1) = (0,0,1) + (x,0,x) +$
$+ (0, y, \frac{1}{2}y) = (0,0,1) + x(1,0,1) + y(0,1,\frac{1}{2})$

Hence the plane goes through $(0,0,1)$ and is spanned by the vectors $(1,0,1)$ and $(0,1,\frac{1}{2})$



(b)



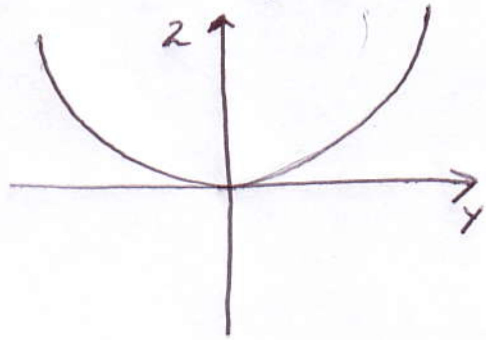
The portion of the graph is a parallelogram with vertices $(1,1,f(1,1))$, $(2,1,f(2,1))$, $(2,4,f(2,4))$, and $(1,4,f(1,4))$

since $f(1,1)=\frac{5}{2}$, $f(2,1)=\frac{7}{2}$, $f(2,4)=5$, $f(1,4)=4$.

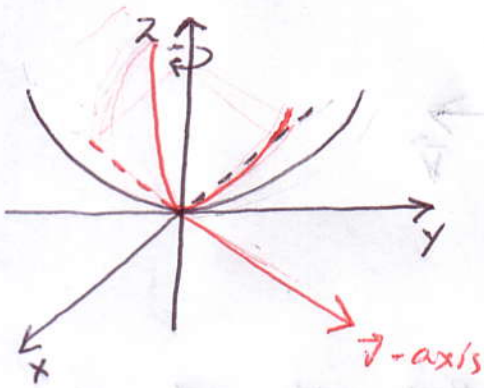
the vertices are $(1,1,\frac{5}{2})$, $(2,1,5)$, $(2,4,\frac{7}{2})$, and $(1,4,4)$

Functions obtained by rotation of a function of a single variable.

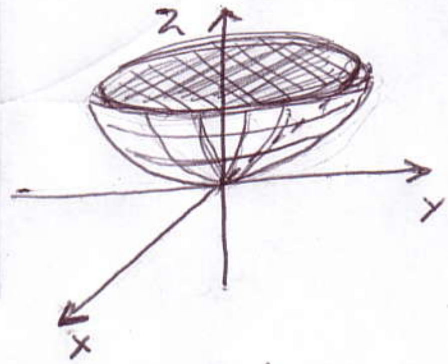
Consider the function $z=f(y)=y^2$. This is a function of a single variable whose graph (drawing of graph) is



What is the explicit formula of the function $g(x,y)$, whose graph is obtained by rotating f about the z-axis?

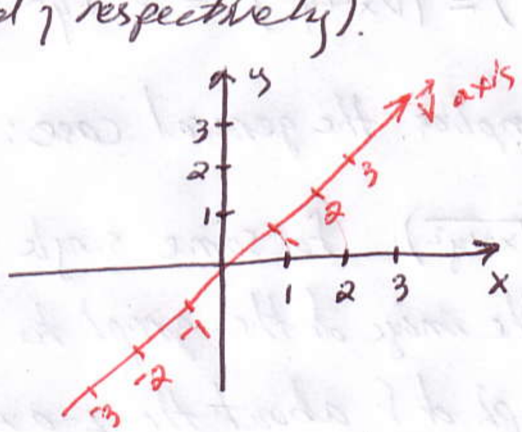


rotation of f about z-axis

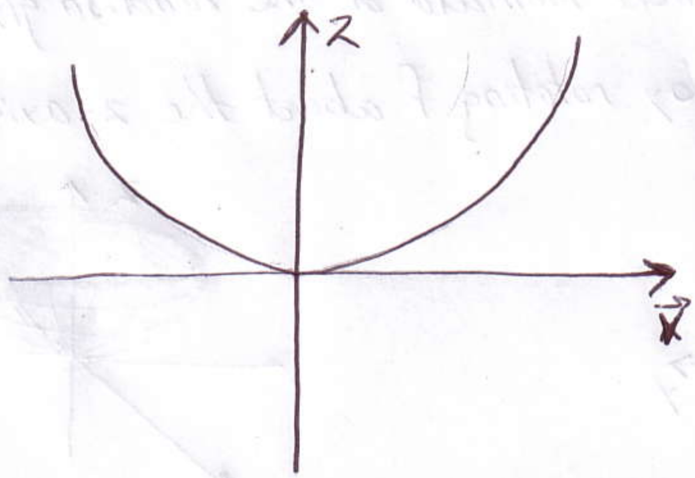


'graph' of $g(x,y)$

To answer this question, recall that any vector $\vec{v} \in \mathbb{R}^2$ generates a number line (the x and y axis are just the number lines generated by the vectors i and j respectively).



Note that the unit vector in the direction of $\vec{v}$ is $\frac{\vec{v}}{\|\vec{v}\|}$. Also observe that if θ is the angle between i and $\frac{\vec{v}}{\|\vec{v}\|}$, then $\frac{\vec{v}}{\|\vec{v}\|} = (\cos\theta, \sin\theta)$ (why?). If r is any number on the $\vec{v}$-axis, then $f(r) = g(r\cos\theta, r\sin\theta)$ because the graph in the z-$\vec{v}$ plane is the same as the graph in the y-z plane:



In particular, $f(r) = g(r\cos\theta, r\sin\theta)$ implies that g depends only on the size of the vector and not on its direction:

$$f(r) = f\left(\sqrt{r^2\cos^2\theta + r^2\sin^2\theta}\right) = f\left(\sqrt{(r\cos\theta)^2 + (r\sin\theta)^2}\right) = f\left(\sqrt{x^2+y^2}\right)$$
$$= g(x,y) \qquad (1)$$

Hence $g(x,y) = f(\sqrt{x^2+y^2}) = (\sqrt{x^2+y^2})^2 = x^2+y^2$

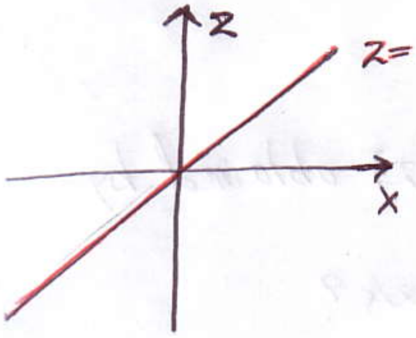
Notice that property (1) implies the general case:

If a function $g(x,y) = f(\sqrt{x^2+y^2})$ for some single variable function f, then the graph of g (or the image of the graph to be precise) is obtained by rotating the graph of f about the z-axis.

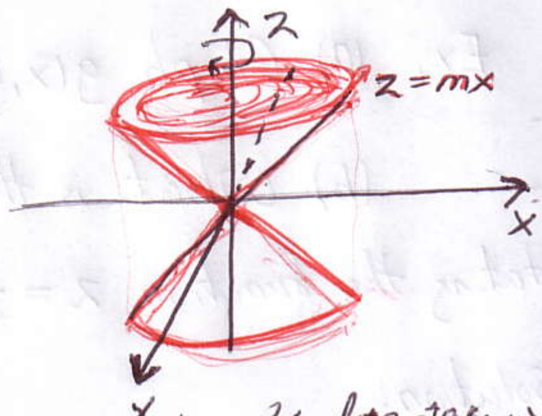
Ex. What is the explicit formula of the function $g(x,y)$, if it is obtained by rotating the function $f(x) = mx$, $m>0$ about the z-axis?

Solution:

The graph of f is a slanted line with positive slope that intersects the x and y-axis at the origin. The graph of g is, consequently, the hour-glass shape. Both graphs are displayed below.



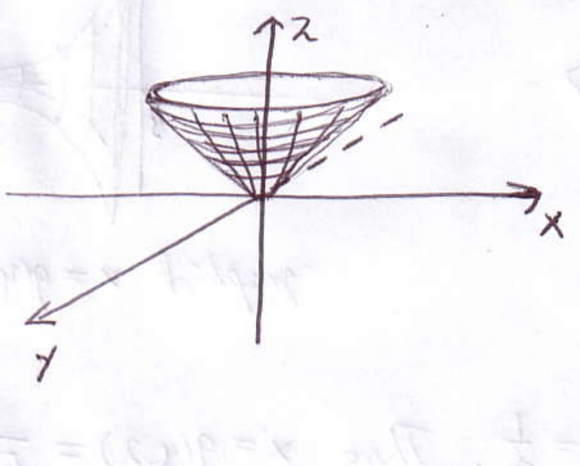
$\Longrightarrow$



The graph of g is called a "right circular cone," because if we make the restriction $z = f(x)$, $x \geq 0$, we obtain a cone with its nose at the origin. By the property of function rotation (1)

$g(x,y) = f(\sqrt{x^2+y^2}) = m\sqrt{x^2+y^2}$

Remark: Strictly speaking, we must require that $x \geq 0$ (or $x \leq 0$) to insure that the function $z = f(x)$ generates a function $g(x,y)$ by means of rotation about the z-axis. For instance, if we rotate $z = mx$, $m > 0$ about the z-axis, we obtain the hour-glass shape drawn above. On the other hand, if $x \geq 0$ then we obtain the graph of $g(x,y) = m\sqrt{x^2+y^2}$



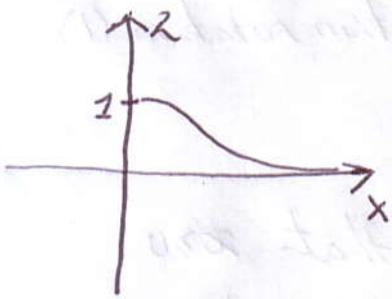
Ex. (a) Graph $g(x,y) = e^{-x^2-y^2}$

(b) What is the equation of $g(y,z)$ obtained by rotating the function $z = \frac{1}{x}$ about the x-axis?

Solution:

(a) Notice that $g(x,y) = e^{-(x^2+y^2)} = e^{-(\sqrt{x^2+y^2})^2}$

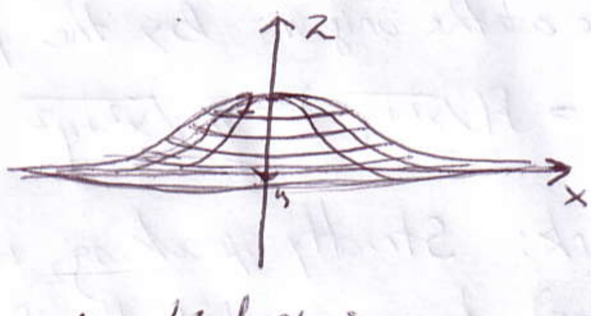
let $f(x) = e^{-x^2}$, then $g(x,y) = f(\sqrt{x^2+y^2})$. Hence, g has a graph that is obtained by rotating $z = e^{-x^2}$ about the z-axis.



'graph' of $f(x)$

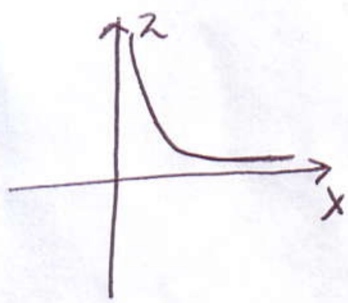
$x \geq 0$

$\Rightarrow$



'graph' of $g(x,y)$.

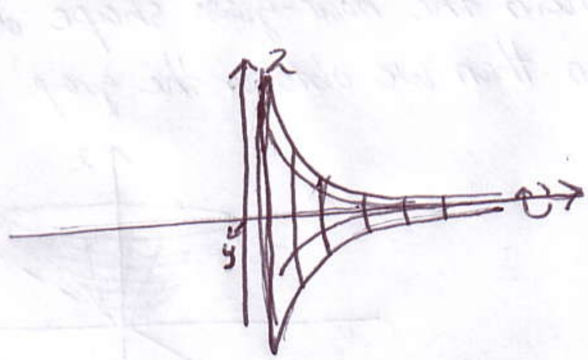
(b) The graph of $g(y,z)$ can be seen below:



'graph' of $z = \frac{1}{x}$

$x > 0$

$\Rightarrow$



'graph' of $x = g(y,z)$

Notice that $z = \frac{1}{x} \equiv x = \frac{1}{z}$. Thus $x = g(y,z) = \frac{1}{\sqrt{y^2+z^2}}$ (why?)

Comprehension check:

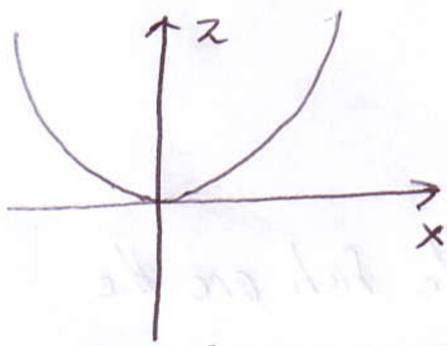
(a) Graph $A(x,y) = \tan(\sqrt{x^2+y^2})$; $x^2+y^2 < \left(\frac{\pi}{2}\right)^2$

(b) Graph $B(x,y) = \frac{1}{1+x^2+y^2}$

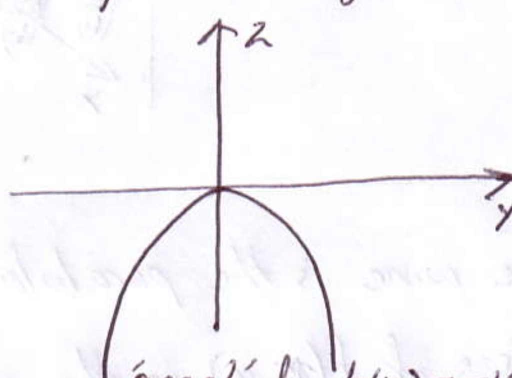
(c) Graph $C(x,z) = 2\sqrt{x^2+z^2}$

Graphs of functions of the form $f(x,y) = g(x) + h(y)$

Consider the function $f(x,y) = x^2 - y^2$ and notice that it is the sum of two functions $g(x) = x^2$ and $h(y) = -y^2$ of a single variable.



'graph' of $g(x) = x^2$



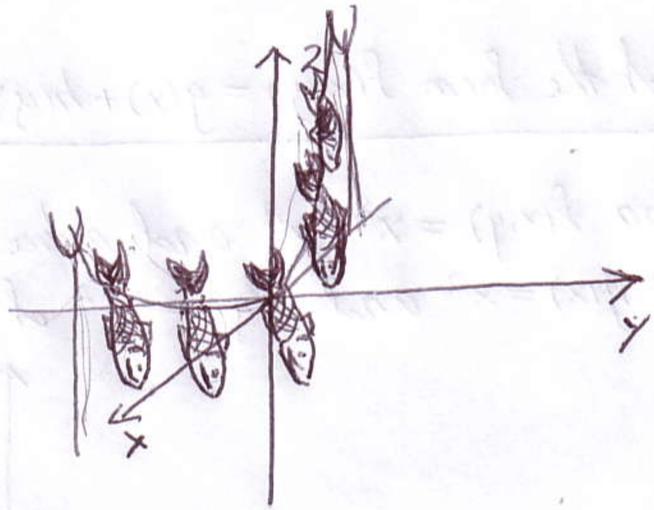
'graph' of $h(y) = -y^2$

Observe that $f(0,y) = -y^2 = h(y)$ and $f(x,0) = x^2 = g(x)$.

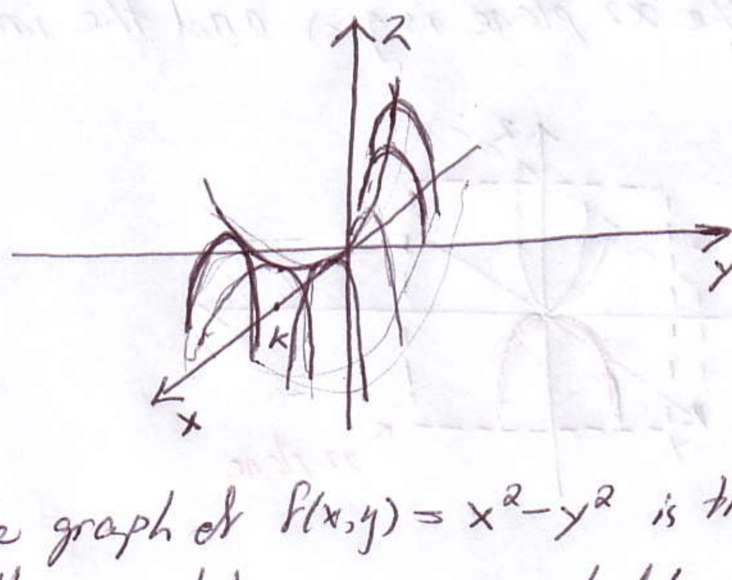
Thus, the image of f in the xz plane is $g(x)$ and the image of f in the yz plane is $h(y)$.

Notice that the xz plane is also the plane $y=0$. Similarly, the yz plane is the plane $x=0$.

We can think of the graph of f as a 'fish wire' ladden with fish that we are trying to smoke (All Russians know this goes well with vodka!).



The wire is the parabola $g(x)=x^2$ and the fish are the images of $f(x,y)$ in the plane $x=k$. That is, a fish with x-coordinate k is the graph of $f(k,y)=k^2-y^2=k^2+h(y)$. This graph is a translation of the graph $h(y)$ up by k^2 units.



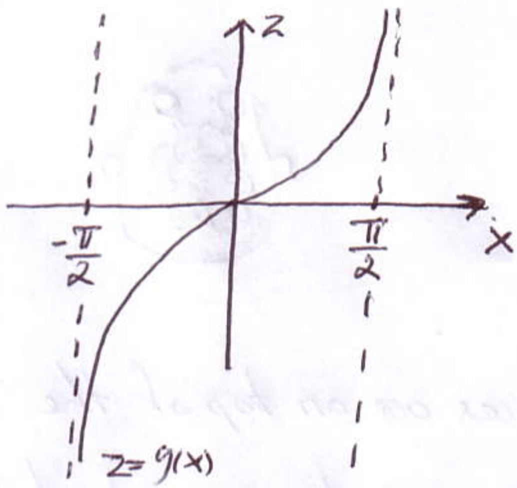
That is, the graph of $f(x,y)=x^2-y^2$ is the graph of a parabula from which other parabolas are suspended. (see comp. generated pic).

In general, a function of the form $f(x,y) = g(x) + h(y)$ can be visualized as 'fish on the wire' where the wire is $g(x)$ and the fish is $h(y)$. (Why?)

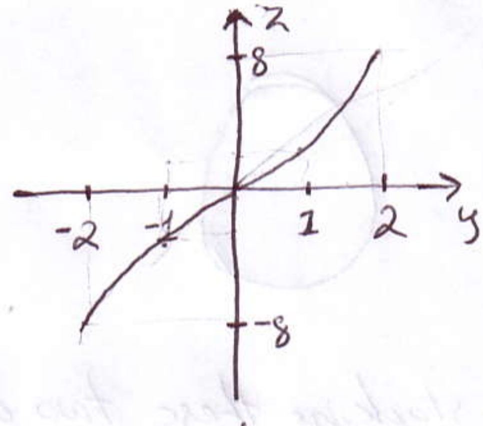
Ex. Graph $f(x,y) = \tan x + y^3$, $\frac{-\pi}{2} < x < \frac{\pi}{2}$ $-2 \le y \le 2$

Solution:

Notice that $g(x) = \tan x$ and $h(y) = y^3$. Hence

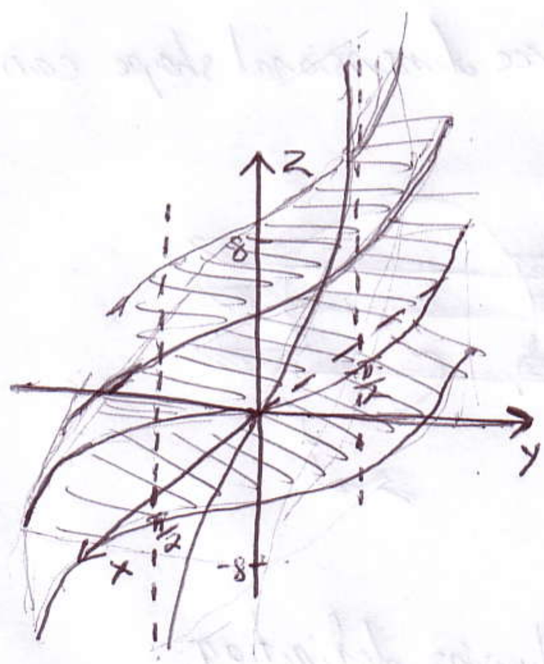


$z = g(x)$



$z = h(y)$

So



$z = f(x,y)$

(see comp. generated pic)

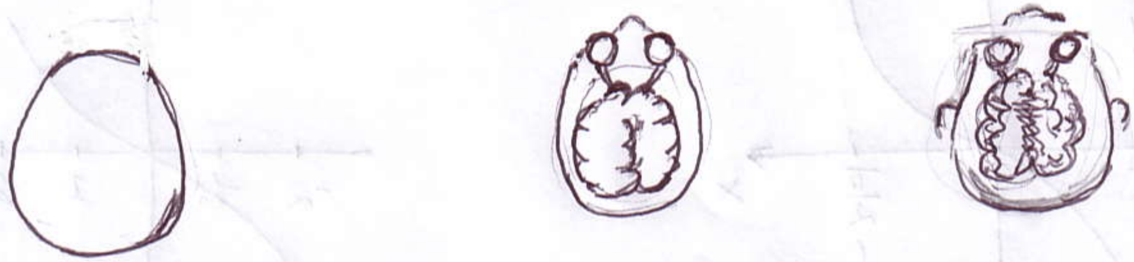
Comprehension check:

Describe the graph of $f(x,y) = \cos x + \sin y$.

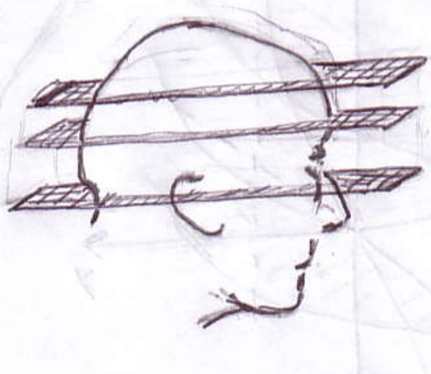
(It is a bit difficult to draw but you can see it in your mind's eye, can't you?)

Level Curves

Have you seen MRI scans of the brain?



By stacking these two dimensional slices one on top of the other, the original three dimensional shape can be reconstructed.



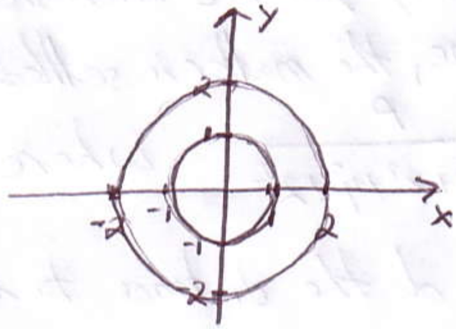
This motivates the following definition.

Def: Let $f: \mathcal{U} \subset \mathbb{R}^2 \to \mathbb{R}$ and let $c \in \mathbb{R}$. Then the level curve of value c is $\{(x,y) : f(x,y) = c\}$.

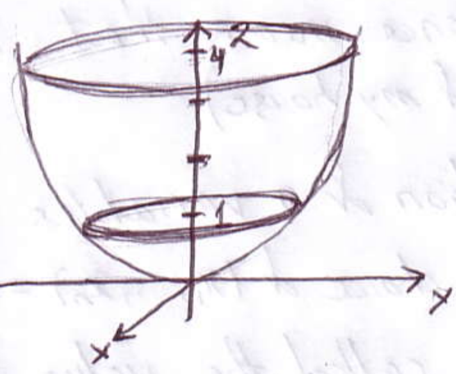
Ex. Graph $f(x,y) = x^2+y^2$.

Solution: We have seen one technique of graphing this function. We'll do this now using level curves.

Notice that $f(x,y) = c \Rightarrow x^2+y^2 = c$. Hence, if $c<0$ the set $L_c = \{(x,y) : f(x,y) = c\} = \emptyset$. If $c>0$ $x^2+y^2=c$ is the equation of a circle centered at the origin with radius $\sqrt{c}$.



Level curves $x^2+y^2=1$ & $x^2+y^2=4$



Comprehension check: Compute and describe the level curves of $f(x,y) = (x^2+1)y$

Functions of n-variables

Many functions of practical interest depend on more than two variables. In fact, computer technology allows consideration and analysis of functions of many thousands of variables. Such functions arise naturally in fields like physics, chemistry, and economics as well as personal matters like choosing a PhD program.

PhD programs want to know why you are interested in this particular program over another. For me, the matter is settled by means of the function $A(x,y,z,p) = \dfrac{p}{1+x^2+y^2+z^2}$ where p is prestige

and $x^2+y^2+z^2$ is the square of the distance to my home.

This functions tells a lot about my personality, I like to attend a well recognized school, but I do not like to travel (the +1 added to the distance means that I don't mind walking beyond the treshold of my house)

Def: A scalar valued function of n variables $x_1, \dots, x_n$ is a rule f, that associates with each choice of $(x_1, \dots, x_n) = \vec{x}$ a single real number $f(\vec{x}) = f(x_1, \dots, x_n)$ called the value of f at $(x_1, \dots, x_n) = \vec{x}$. The set of ordered n-tuples $(x_1, \dots, x_n) = \vec{x}$ for which the rule is specified is the domain of the function. We sometimes denote the domain of f with the letter '$\mathcal{U}$'. Thus, we write $f: \mathcal{U} \subset \mathbb{R}^n \to \mathbb{R}$ to explicitly state that f is a real valued function with domain $\mathcal{U}$.

Ex. Given vectors $\vec{a}_1, \vec{a}_2, \dots \vec{a}_{n-1} \in \mathbb{R}^n$, Define $f: \mathbb{R}^n \to \mathbb{R}$ by $f(\vec{x}) = \det\begin{pmatrix} \vec{x} \\ \vec{a}_1 \\ \vdots \\ \vec{a}_n \end{pmatrix}$. Then f is a function of n-variables with domain $\mathbb{R}^n$.

Graph of a Function of n-Variables

As we mentioned before, the graph of a function is a set, whereas the drawing of the graph is a pictorial representation of this set. We now state this precisely.

Def: Let $f: \mathcal{U} \subset \mathbb{R}^n \to \mathbb{R}$. Then the graph of f is a subset of $\mathbb{R}^{n+1}$ consisting of all the points

$$(x_1, \dots, x_n, f(x_1, \dots, x_n)) \text{ in } \mathbb{R}^{n+1} \text{ for } (x_1, \dots, x_n) \in \mathcal{U}$$

In symbols graph of $f = G_f = \{(x_1, \dots, x_n, f(x_1, \dots, x_n)) \in \mathbb{R}^{n+1} : (x_1, \dots, x_n) \in \mathcal{U}\}$.

Level sets and surfaces

Just like with level curves, we may consider sets of the form $L_c = \{(x,y,z) \in \mathcal{U} : f(x,y,z) = c\}$ for functions of the form $f: \mathcal{U} \subset \mathbb{R}^3 \to \mathbb{R}$. In this case, the sets L_c are known as level surfaces, because they can be represented pictorially as the surfaces of 3-D objects.

For example, suppose $f(x,y,z) = x^2+y^2+z^2$. Then $L_c = \emptyset$ for all $c<0$. If $c>0$, for instance $c=1$, then $L_c = \{(x,y,z) : x^2+y^2+z^2 = c\}$. In words, L_c is the set of all points, such that the square of the distance from each point to the origin is c. Hence L_c may be identified geometrically with the surface of a ball of radius $\sqrt{c}$ that is centered at the origin. In particular, L_c is a sphere.

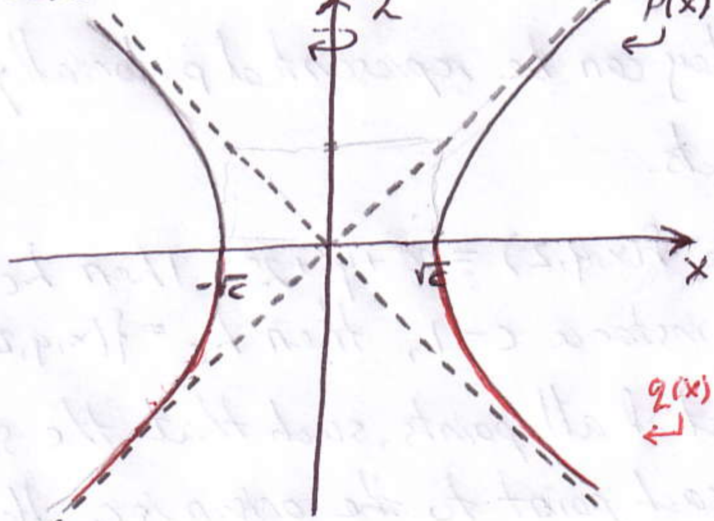
While we cannot represent the graphs of functions of 3 variables pictorially, we can still infer something about the geometry of the 4-D graphs from its 3-D 'cross sections'.

Ex. Consider $f(x,y,z) = x^2+y^2-z^2$. The level surfaces of f are identified by equations of the form $x^2+y^2-z^2 = c$ where c is a constant.

(i) If $c>0$ $x^2+y^2-z^2 = c \Rightarrow x^2+y^2-c = z^2 \Rightarrow$ $z = \pm\sqrt{x^2+y^2-c}$. Hence L_c is a union of the graphs of the function $z = g(x,y) = \sqrt{x^2+y^2-c}$ and $z = h(x,y) = -\sqrt{x^2+y^2-c}$.

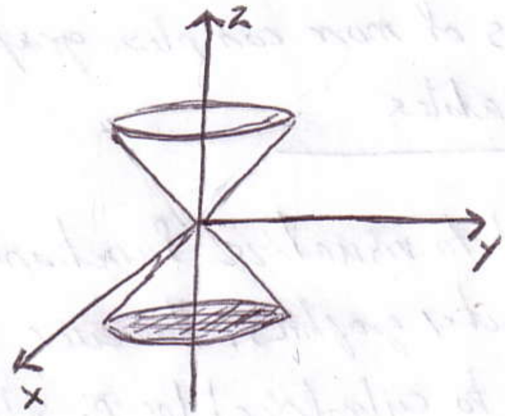
Note that $g(x,y)$ is obtained by rotating the function $z = p(x) =$ $= \sqrt{x^2-c}$ about the z-axis. Similarly $h(x,y)$ is obtained by rotating the function $z = q(x) = -\sqrt{x^2-c}$ about the z-axis.

Calc I techniques allow us to draw $p(x)$ and $q(x)$. Their graphs are displayed below:

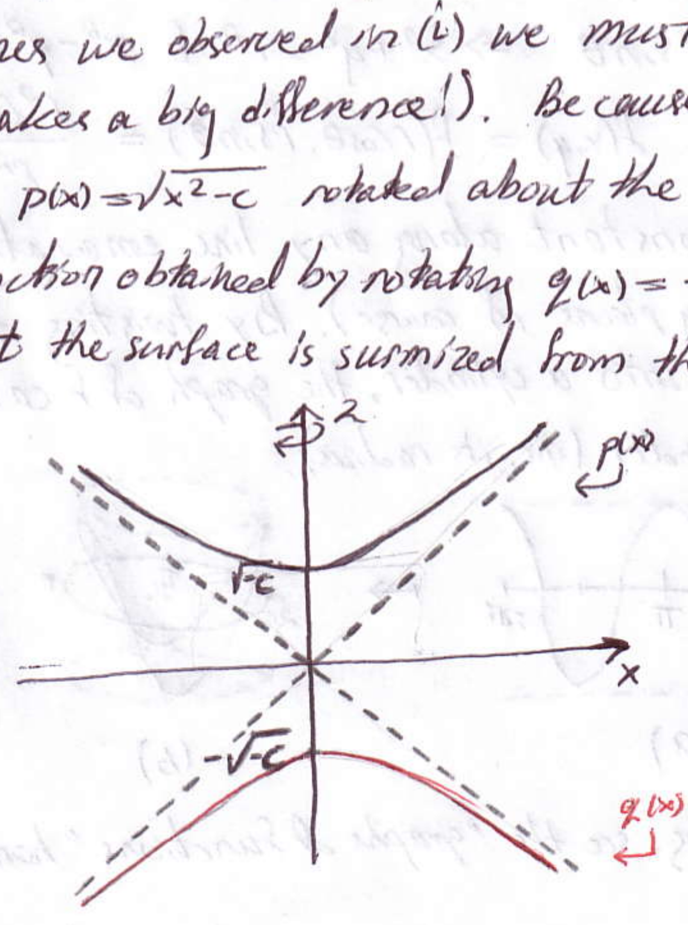


From here, L_c, the set of all points satisfying the equation $x^2+y^2-z^2 = c>0$ is easily visualized. (You can also see it on p 104)

(ii) If $c=0$ $x^2+y^2-z^2=0 \Rightarrow x^2+y^2=z^2 \Rightarrow$ $z=\pm\sqrt{x^2+y^2}$. This is simply the hour-glass shape of a circular cone.



(iii) Finally, if $c<0$, $x^2+y^2-c=z^2 \Rightarrow z \geq \sqrt{-c}$ or $z \leq -\sqrt{-c}$ (why?)
The graph of h_c is the union of the graphs $g(x,y)=\sqrt{x^2+y^2-c}$ and $h(x,y)=-\sqrt{x^2+y^2-c}$. Although g, and h seem to have similar formulas to the ones we observed in (i) we must remember that $c<0$ (and that makes a big difference!). Because $g(x,y)$ can be viewed as $p(x)=\sqrt{x^2-c}$ rotated about the z axis, and since $h(x,y)$ is a function obtained by rotating $q(x)=-\sqrt{x^2-c}$ about the z axis, we see that the surface is surmized from the drawings below:



More generally, the set $L_c = \{(x_1,\dots,x_n) \in U : f(x_1,\dots,x_n) = c\}$ is called a level set whenever $n \geq 4$.

Pictorial Representations of more complex graphs of functions of two variables

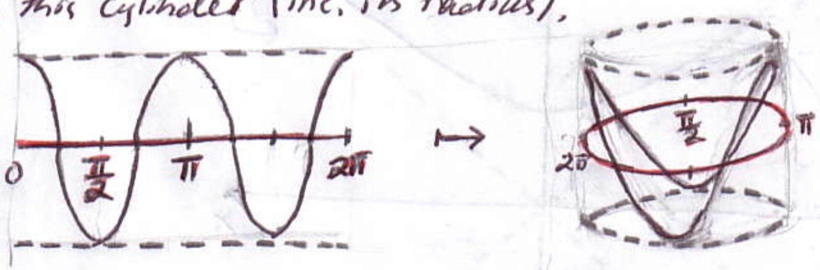
It is generally very hard to visualize functions of two variables without the aid of computer graphics. However, some graphs may be visualized by transitioning to cylindrical (or spherical) coordinates.

Ex. Graph $f(x,y) = \dfrac{x^2-y^2}{x^2+y^2}$.

Solution: This graph is rather hard to visualize in rectangular coordinates. By transitioning to cylindrical coordinates, our task is made considerably simpler.

$x = r\cos\theta \quad y = r\sin\theta \Rightarrow x^2+y^2 = r^2$ & $x^2-y^2 = r^2(\cos^2\theta - \sin^2\theta) =$
$= r^2(\cos 2\theta)$. Hence $f(x,y) = f(r\cos\theta, r\sin\theta) = \dfrac{r^2\cos 2\theta}{r^2} = \cos 2\theta$.

In particular, f is constant along any line emanating from the origin (this line being in the xy plane of course). By twisting the Calc I picture of $\cos 2\theta$, $0 \leq \theta < 2\pi$ into a cylinder, the graph of f can be visualized by expanding this cylinder (i.e. its radius).



(a) (b)

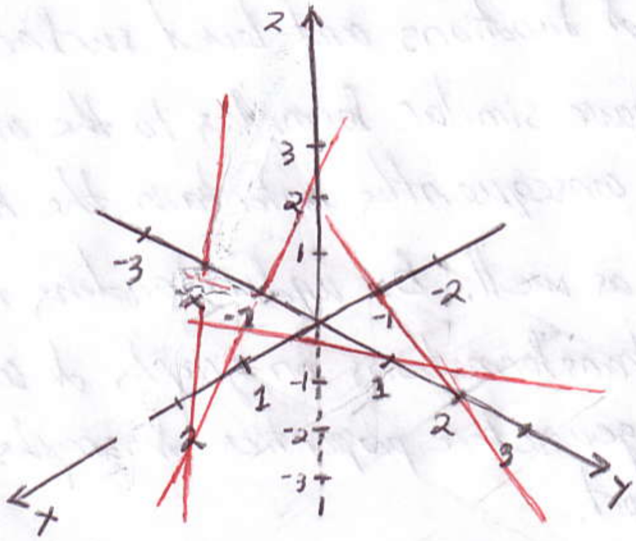
For a 3-D rendering, see the "graphs of functions" handout.

Ex. Graph $f(x,y) = xy$

Solution:

<u>Method 1</u>.

The intersection of the graph of f with the plane $y=c$ is $z = xc$. This is a line with slope c. Several such intersections are drawn below



Thus, the graph of $f(x,y) = xy$ looks like a rectangular sheet of paper that is about to be torn. (See computer-generated image).

<u>Method 2</u>

Notice that $f(x,y) = f(r\cos\theta, r\sin\theta) = r^2\sin\theta\cos\theta = \frac{1}{2}r^2\sin 2\theta$. It follows that a cylindrical cut-off of the graph of f will be a shape similar to drawing (b) of the previous excersise stretched over the parabola $z = \frac{1}{2}r^2$. (See computer-generated image).

Comprehension check: Use cylindrical coordinates to visualize

(a) $f(x,y) = \frac{xy}{x^2+y^2}$

(b) $g(x,y) = x^2 - y^2$

Graphing by means of coordinate transformations

Visualizing the graphs of functions and level surfaces is an arduous task. Some functions have similar formulas to the ones we already studied and we might consequently entertain the hope that their graphs are similar as well. By understanding the effects of spatial and coordinate transformations on graphs of arbitrary functions, we can often unravel geometric properties of graphs without expending too much effort.

For instance, suppose we wished to graph $f(x,y) = (x-1)^2 + (y+3)^2 + 2$. Under close examination $f(x,y)$ appears to have a formula similar to $g(x,y) = x^2+y^2$. We have seen before that the graph of g is obtained by rotating $p(x) = x^2$, $x \geq 0$ about the z-axis. In particular, the graph of g is known to us. So, in order to visualize f, let's make its formula like that of g.

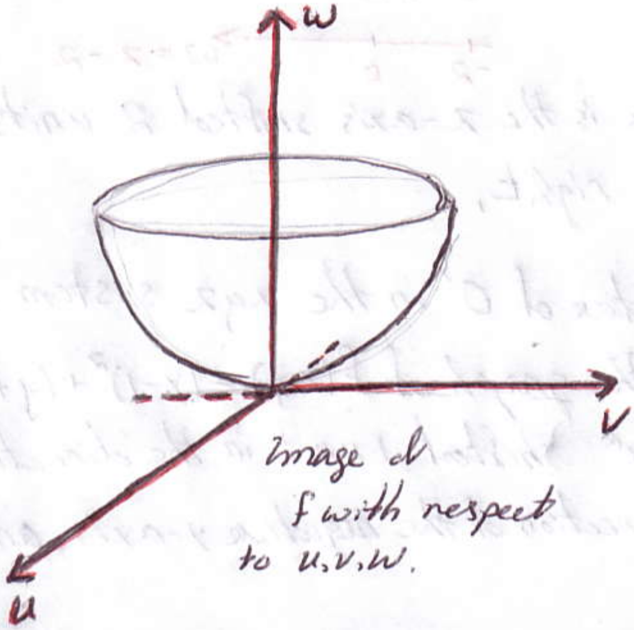
$$z = f(x,y) = (x-1)^2 + (y+3)^2 + 2$$

$$z = (x-1)^2 + (y+3)^2 + 2$$

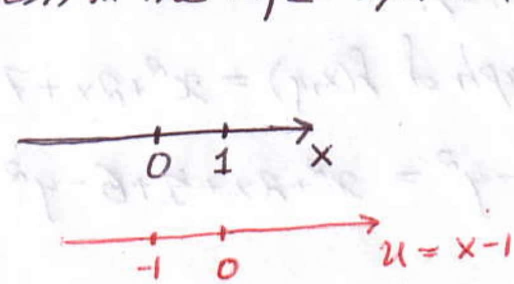
$$z - 2 = (x-1)^2 + (y+3)^2$$

$$w = u^2 + v^2 \qquad \text{where } w = z-2,\ u = x-1,\ v = y+3$$

Thus, in the u,v,w coordinate system the graph is identical to the graph of g in the xyz coordinate system.

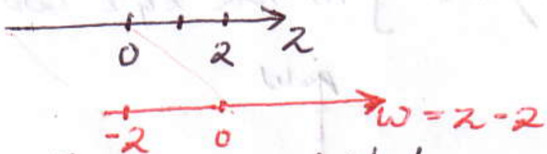


All we must do is determine the coordinates of the origin O' of the uvw coordinate system in the xyz system.



u-axis is the x-axis shifted by 1 unit to the right,

v-axis is the y-axis shifted by 3 units to the left.



w-axis is the z-axis shifted 2 units to the right,

Hence the coordinates of O' in the xyz system are $(1, -3, 2)$, which means that the graph of $f(x,y) = (x-1)^2 + (y+3)^2 + 2$ is the graph of $g(x,y) = x^2 + y^2$ shifted 1 unit in the direction of the positive x-axis, 3 units in the direction of the negative y-axis, and 2 units in the positive z-axis.

In general, $z = f(x,y) \mapsto z = f(x+a, y+b) + c$ is a shift that 'relocates' the graph of the function $z = f(x,y)$ onto the new origin O' with xyz coordinates $(-a, -b, c)$.

Ex. Describe the graph of $f(x,y) = x^2 + 2x + 7 - y^2$.

Solution: $x^2 + 2x + 7 - y^2 = x^2 + 2x + 1 + 6 - y^2 = (x^2 + 2x + 1) - y^2 + 6 =$
$= (x+2)^2 - y^2 + 6$

So $z = f(x,y) = (x+2)^2 - y^2 + 6$ or $z - 6 = (x+2)^2 - y^2$

letting $w = z - 6$, $u = x + 2$, and $v = y$, we get $w = u^2 - v^2$ which is the equation of a hyperbolic parabuloid in uvw coordinates, where the origin of the u-v-w system is $(-2, 0, 6)$. Thus the shift is 2 unit in the negative x-axis, 0 units in the y-axis, and 6 units in the positive z-axis.

Spatial transformations can also be used to visualize level surfaces.

Ex. the equation $(x-1)^2+(y+3)^2+(z+2)^2=9$ is identical to $u^2+v^2+w^2=9$ which is the equation of a sphere with radius 3 with center $O'=(0',0',0')$ in the u-v-w coordinate system.
Hence, the original equation identifies a sphere with radius 3 and center $O'=(0',0',0')\equiv(1,-3,-2)$ in the x-y-z coordinate system.

Ex. Recall that the equation $z^2=x^2+y^2$ has a graph that is the union of the graphs $g(x,y)=\sqrt{x^2+y^2}$ and $h(x,y)=-\sqrt{x^2+y^2}$.
Hence $z^2=x^2+y^2$ identifies the hour-glass shape made of two cones (see page 11)
It follows that $(z-5)^2=(x+1)^2+(y-4)^2$ is the hour-glass shape made of two cones. The tips of these cones touch at the coordinates $(-1,4,5)$

Transformations that stretch space

We have seen the effects of solid translations $(x,y,z)\mapsto(x-a,y-b,z-c)$.
They are called solid translations, because they shift space (and objects in space) without stretching or compressing it.
Many transformations stretch space and change the volume of solids.
Let's see a few examples where this happens.

Ex. Graph $f(x,y)=9x^2+4y^2$

Solution: $z = 9x^2 + 4y^2 \Rightarrow (3x)^2 + (2y)^2$.

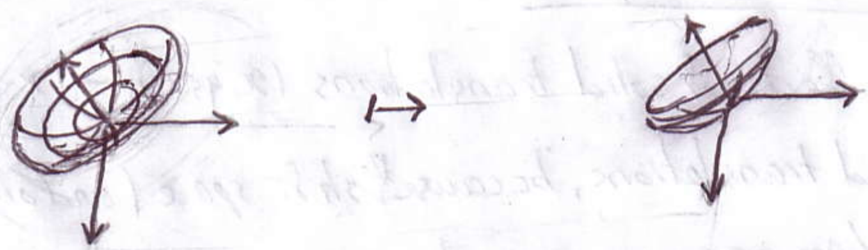
Thus, by replacing $3x$ with u, $2y$ with v, and z with w, we get $w = u^2 + v^2$. In the u-v-w coordinate system the last equation identifies a geometric shape, known as circular paraboloid (shown on page 25)

What might be the shape of the graph in the x-y-z system?

the u-axis is obtained from the x-axis by compressing the x-axis by a factor of 3

Similarly, you should verify that the v-axis is generated from the y-axis by compressing it by a factor of 2.

Thus, we can visualize the graph of $z = 9x^2 + 4y^2$ as a squeezed bowl $z = x^2 + y^2$:



(See hand-out on graphs)

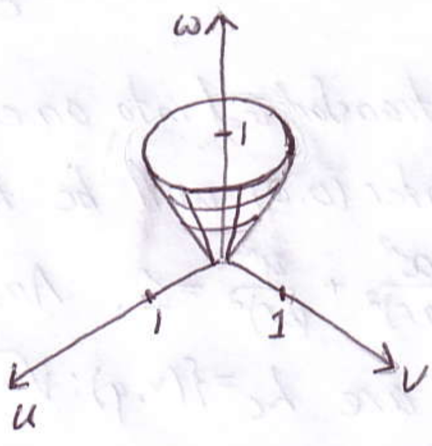
In general, if $a,b>0$, the transformation $f(x,y) \mapsto f(ax, by)$ stretches the x and y-axis by a factor of $\frac{1}{a}$ and $\frac{1}{b}$ respectively (Can you explain why?)

Ex. Describe the graph of $f(x,y) = \sqrt{\frac{x^2}{4} + \frac{y^2}{9}}$

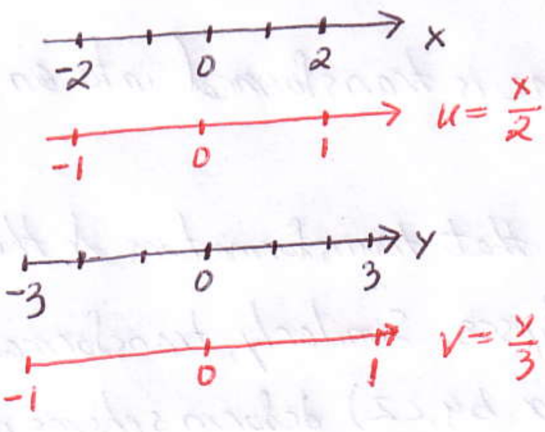
Solution:

$$z = f(x,y) = \sqrt{\left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2} = \sqrt{u^2+v^2} \quad \text{where } u = \frac{x}{2},\ v = \frac{y}{3}$$

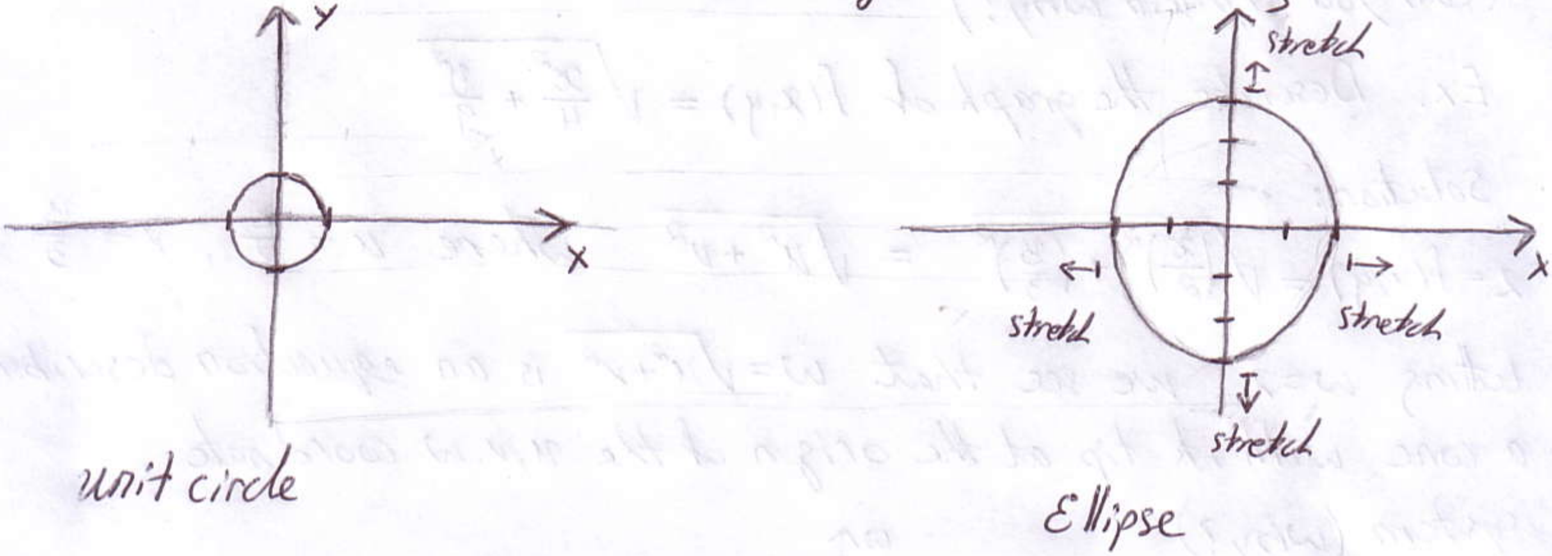
Letting $w=z$, we see that $w=\sqrt{u^2+v^2}$ is an equation describing a cone with its tip at the origin of the u,v,w coordinate system (why?)



By the general statement above u is the x-axis stretched by a factor of $\frac{1}{a} = \frac{1}{(\frac{1}{2})} = 2$ and v is the y-axis stretched by a factor of $\frac{1}{b} = \frac{1}{(\frac{1}{3})} = 3$. Since this fact is very important, let's see the argument yet again.



Note that only the x and y-axes are stretched. To see the effects of this stretching on a 2-D object like a unit circle, for example consider the diagram below:



In fact, any circle will be transformed into an ellipse: More concretely, a circle of radius r and center $(0,0)$ will be transformed into an ellipse with equation $\frac{x^2}{(ar)^2} + \frac{y^2}{(br)^2} = 1$. And since the level curves of $g(x,y) = \sqrt{x^2+y^2}$ are $L_c = \{(x,y): \sqrt{x^2+y^2} = c,\ c \geq 0\} =$
$= \{(x,y): x^2+y^2 = c^2\} \equiv$ circles with center $(0,0)$ and radius c,
it follows that the level curves of $f(x,y) = \sqrt{\left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2}$ are ellipses with equation $\frac{x^2}{4c^2} + \frac{y^2}{9c^2} = 1$ (i.e. semi major axes $3c$ and semi minor axes $2c$)

In particular, the cone is transformed into an elliptic (or squeezed) cone.

* It is interesting to note that transformations of the form $(x,y) \mapsto (ax, by)$ deform circles into ellipses. Similarly, transformations on 3D spaces of the form $(x,y,z) \mapsto (ax, by, cz)$ deform spheres into ellipsoids.

or (American) football head trauma projectiles.

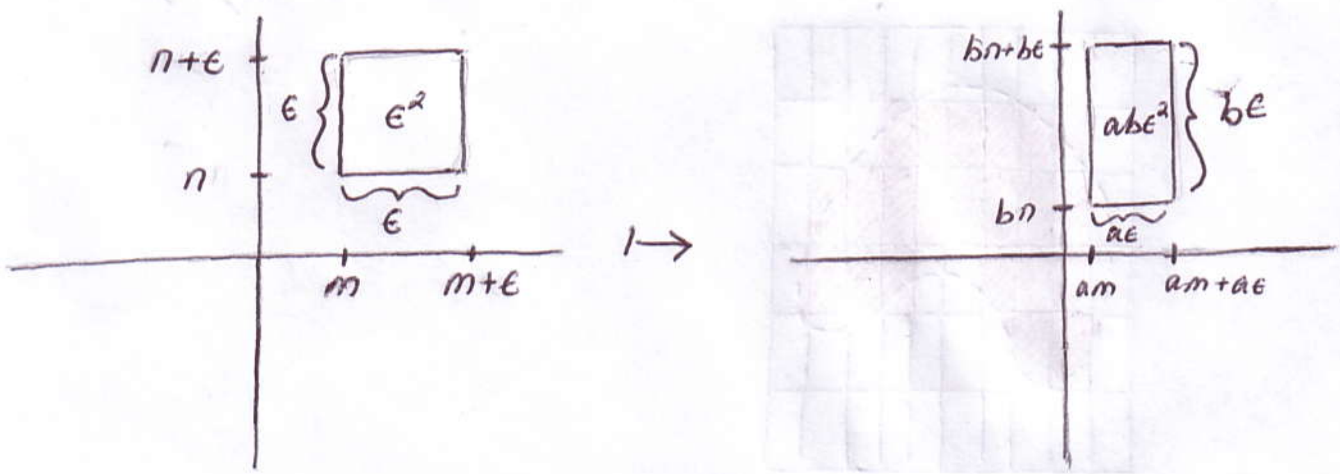
Comprehension check:

Describe the shape of the surface $\frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{16} = 1$.

Area of an ellipse and Volume of an ellipsoid

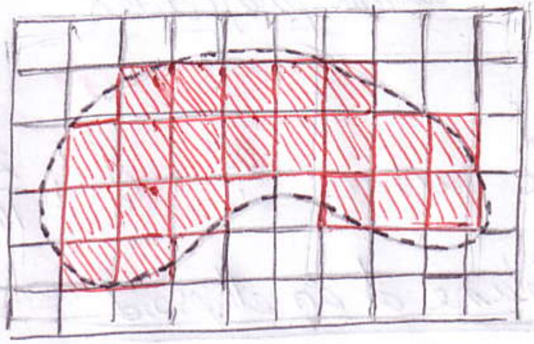
Let us consider what a transformation of the form $(x,y) \mapsto (ax, by)$ will do to a square with sides ϵ? Suppose this square is positioned with one of its edges at the point (m,n). Then the remaining edges are positioned at $(m, n+\epsilon)$, $(m+\epsilon, n)$, and $(m+\epsilon, n+\epsilon)$.

Our transformation maps (m,n) to (am, bn), $(m, n+\epsilon) \mapsto (am, bn+b\epsilon)$, $(m+\epsilon, n) \mapsto (am+a\epsilon, bn)$, and $(m+\epsilon, n+\epsilon) \mapsto (am+a\epsilon, bn+b\epsilon)$.



In particular, every square of side ϵ is stretched into a rectangle with sides $a\epsilon$ and $b\epsilon$. The area of the square is multiplied by the factor ab to produce the area of the rectangle.

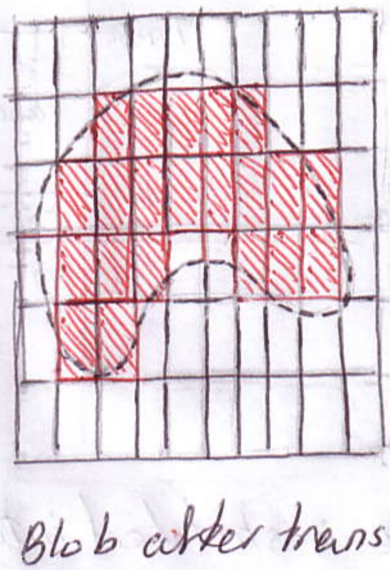
How is this useful? Consider a 'blob' with area A. We can think of this blob as a structure consisting of small squares or pixels of size ϵ. When ϵ is small, the 'pixel structure' closely resembles the blob.



-- Blob with area A

— Pixel Structure approximating blob.

Under the transformation $(x,y) \mapsto \left(\left[\tfrac{1}{a}\right]x, \left[\tfrac{1}{b}\right]y\right)$, the square pixel structure (whose area closely resembles the area of the blob for small ϵ) becomes a rectangular structure, whose area is ab times the area of the square pixel structure (why?)



-- Blob after transformation

— Pixel Structure after transformation.

The new pixel structure will be a good approximation to the modified blob for small ϵ. Hence

A = area of blob $\approx$ area of pixel structure $\mapsto$ area of transformed pixel structure $= ab \cdot$(area of original pixel structure) $\approx ab \cdot A$.

In particular, the area enclosed by any curve in the xy coordinate system is modified (scaled) by a factor of ab under the transformation $(x,y) \mapsto \left(\left[\frac{1}{a}\right]x, \left[\frac{1}{b}\right]y\right)$.

Let's put this observation to use.

Ex. Find the area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Solution: consider the equation of the unit circle $x^2+y^2=1$. We know that its area is π. Under the transformation $(x,y) \mapsto (\frac{1}{a}x, \frac{1}{b}y)$ this circle is mapped into the ellipse $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$ and the area is scaled by a factor of ab. Hence the area of the ellipse πab.

* Notice that when $a=b$ the equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is the same as $\frac{x^2}{a^2} + \frac{y^2}{a^2} = 1$ or $x^2+y^2=a^2$, which is a circle with radius a and area πa^2.

Comprehension check: Find the volume of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$. Justify your answer.

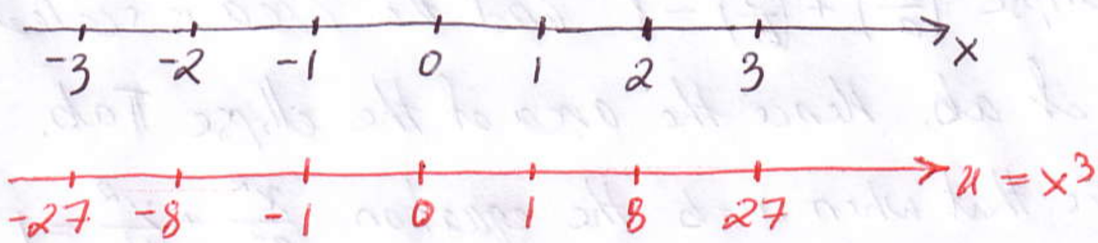
Uniform v.s. non-uniform stretch of space

We have seen that transformations like $(x,y) \mapsto (x+a, y+b)$ simply shift space without distorting it, whereas transformations like $(x,y) \mapsto (ax, by)$ distort each section of the 2-D space as if it were a rubber band. (The same observations can be made about transformations of 3-D and even n-D space).

Transformations of the form $(x,y) \mapsto (ax, by)$ stretch space uniformly (evenly) in all directions, because a square placed anywhere in this space will be stretched by the same factor. There are, however, unimaginably many transformations that distort space in a curious and non-uniform manner.

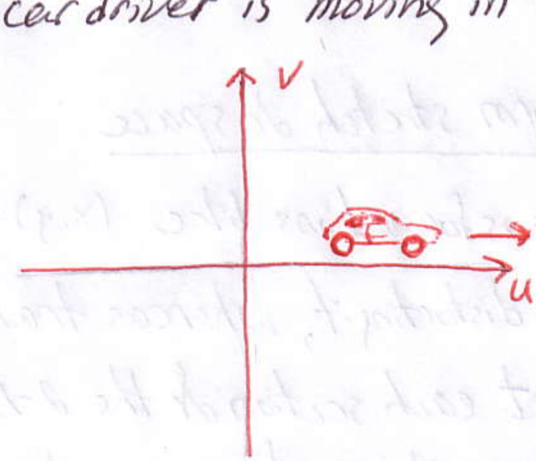
Ex, Consider the transformation $(x,y) \mapsto (x^3, y)$.

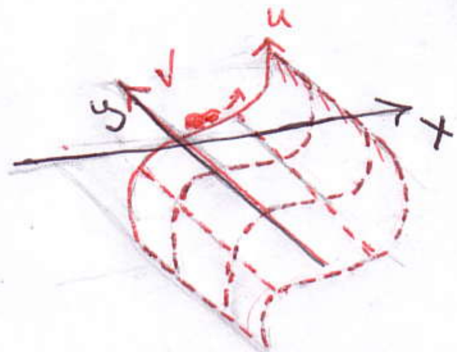
let $u = x^3$ and $v = y$. What kind of deformation produces the u-axis out of the x-axis?



Notice that the u-axis is being compressed more and more as we are moving away from 0.

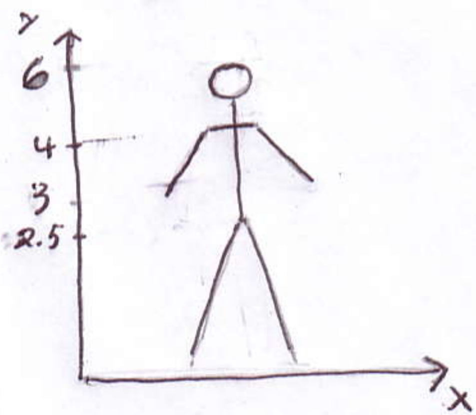
What do you think will happen to the race car driver from the perspective of a person standing in the xy coordinate system if the race car driver is moving in the positive u-axis in the uv space?





The uv system appears to be curved with respect to the xy coordinate system. A car traveling along the u-axis with constant speed will appear to slow down and its length compressed towards singularity (point).

Comprehension check: Draw the image of the person below under the transformation $(x, y) \to (x, y^2)$



Does this transformation stretch space uniformly?