Cross Product And Associativity

Given three vectors $\vec{a}$, $\vec{b}$, $\vec{c}$, is the triple product $(\vec{a}\times\vec{b})\times\vec{c}$ the same vector as $\vec{a}\times(\vec{b}\times\vec{c})$? To answer this question without performing a tedious computation, it would be helpful to derive a simpler formula for the expression $(\vec{a}\times\vec{b})\times\vec{c}$. To that end, observe that the vector $(\vec{a}\times\vec{b})\times\vec{c}$ is orthogonal to both $(\vec{a}\times\vec{b})$ and $\vec{c}$. Now, each vector that is perpendicular to $(\vec{a}\times\vec{b})$ can be expressed as $s\vec{a}+t\vec{b}$, where s and t are scalars (Why?). To determine the values of s and t that make $s\vec{a}+t\vec{b}$ perpendicular to $\vec{c}$, we solve the equation

$$\vec{c}\cdot(s\vec{a}+t\vec{b})=0.$$

Solving for s in the above equation yields $s=-\frac{\vec{c}\cdot\vec{b}}{\vec{c}\cdot\vec{a}}t$. Thus, any vector of the form $-\frac{\vec{c}\cdot\vec{b}}{\vec{c}\cdot\vec{a}}t\vec{a}+t\vec{b}=\left(\vec{b}-\frac{\vec{c}\cdot\vec{b}}{\vec{c}\cdot\vec{a}}\vec{a}\right)t$ is orthogonal to $(\vec{a}\times\vec{b})$ and $\vec{c}$. Setting $t=(\vec{c}\cdot\vec{a})r$, for some scalar r, yields

$$(\vec{a}\times\vec{b})\times\vec{c}=\left[(\vec{c}\cdot\vec{a})\vec{b}-(\vec{c}\cdot\vec{b})\vec{a}\right]\ r.$$

In fact, it can be shown from the properties of multilinear maps (introduced later on in the course as optional reading material) that r = 1. Hence,

$$(\vec{a}\times\vec{b})\times\vec{c}=\left[(\vec{c}\cdot\vec{a})\vec{b}-(\vec{c}\cdot\vec{b})\vec{a}\right].$$

Now let's compute $\vec{a}\times(\vec{b}\times\vec{c})$:

$$\vec{a}\times(\vec{b}\times\vec{c})=-\left[(\vec{b}\times\vec{c})\times\vec{a}\right]=-\left[(\vec{a}\cdot\vec{b})\vec{c}-(\vec{a}\cdot\vec{c})\vec{b}\right]=(\vec{a}\cdot\vec{c})\vec{b}-(\vec{a}\cdot\vec{b})\vec{c}.$$

Hence $(\vec{a}\times\vec{b})\times\vec{c}=\vec{a}\times(\vec{b}\times\vec{c})$ if and only if $(\vec{c}\cdot\vec{b})\vec{a}=(\vec{a}\cdot\vec{b})\vec{c}$. Thus, the cross product is **not** commutative whenever $(\vec{c}\cdot\vec{b})\neq 0$, $(\vec{a}\cdot\vec{b})\neq 0$, and the vectors $\vec{a}$ and $\vec{c}$ are linearly independent.