HW. # 6

Homework problems are taken from textbook. The problems are color coded to indicate level of difficulty. The color green indicates an elementary problem, which you should be able to solve effortlessly. Yellow means that the problem is somewhat harder. Red indicates that the problem is hard. You should attempt the hard problems especially.

Find the matrices for the following linear transformations.

1. (a) $T(x_1, x_2) = (3x_1 + 5x_2, x_2, -x_1 + 4x_2)$
   (b) $T(x_1, x_2, x_3, x_4) = (x_1 - x_2, x_3 + x_4)$
   (c) $T(x_1, x_2, x_3) = (4x_1 + \frac{x_2}{2} + \frac{x_3}{3}, \frac{x_1}{2} + 3x_2 - \frac{x_3}{4}, \frac{x_1}{3} + \frac{x_2}{4} + 2x_3)$
   (d) $T(x_1, x_2, x_3, \ldots, x_{20}) = x_1 + x_2 + x_3 + \ldots + x_{20}$

2. Classify which functions are linear and which are not
   (a) $T(x_1, x_2) = (2x_1 + x_2, -2x_1 + 5x_2)$
   (b) $T(x_1, x_2) = \frac{x_1^2 x_2}{x_1 + x_2}$
   (c) $T(x, y, z, w) = (x + 3w, -y + 2z + w, xy)$

3. Find the compositions TS and ST whenever they are defined
   (a) $S(x, y) = (2x + y, -2x + 5y)$; $T(x, y) = (\frac{1}{2}x + y, -\frac{1}{2}x + \frac{1}{5}y)$
   (b) $S(x, y, z) = (x + z, y, -z + y)$; $T(x, y, z) = (z, x, y)$
   (c) $S(x, y) = (2x - 3y, x, y, 4x - y)$; $T(x, y, z) = (x, 0)$

4. Find the inverse of the following matrices, if possible
   (a) $\begin{pmatrix} 1 & 3 \\ 1 & 2 \end{pmatrix}$
   (b) $\begin{pmatrix} 4 & -1 \\ -8 & 2 \end{pmatrix}$
   (c) $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

5. Find the inverse of the following matrices, if possible
   (a) $\begin{pmatrix} -2 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -6 \end{pmatrix}$

(b) $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$

(c) $\begin{pmatrix} 1 & 3 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$ (Hint: It suffices to find the inverses of $\begin{pmatrix} 1 & 3 \\ 1 & 2 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. See problem 4)

6. The **norm** of an $m \times n$ matrix A = $\lfloor a_{ij} \rfloor$ is $\|A\| = \sqrt{\sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij}^2}$. Calculate the norm of the following matrices.

(a) $\begin{pmatrix} 3 & 1 \\ 7 & -2 \end{pmatrix}$

(b) $\begin{pmatrix} 6 & 1 & -3 \\ 0 & 6 & 1 \\ 0 & 0 & 6 \end{pmatrix}$

(c) $\begin{pmatrix} 1 & 1 & 0 & 4 \\ -4 & 0 & -1 & -1 \end{pmatrix}$

7. Prove that with the norm of a matrix defined as in exercise 6, $\|Ax\| \le \|A\|\|x\|$. (Hint: Ax = $\sum a_i x_i$, where a_i is the ith column of A. Use triangle inequality and the Cauchy-Schwarz inequality.)

8. For each linear transformation below, determine if it is invertible. If so, find the inverse.

(a) $T(x, y) = (2x + y, x + 5y)$

(b) $T(x, y) = (2x - 3y, -4x + 6y)$

(c) $T(x_1, x_2, ..., x_{10}) = (10x_1, 9x_2, 8x_3, ..., 2x_9, x_{10})$

(d) $T(x_1, x_2, x_3, x_4) = (x_2, x_3, x_4, 0)$

9. Find a matrix for each of the given linear transformations of R^2.

(a) Reflection in the y-axis

(b) Rotation by 90° counterclockwise about the origin

(c) Projection onto the line ax + by = 0

(d) Reflection in the line 4x + 5y = 0 followed by rotation by 90° counterclockwise about the origin. (This one is the hardest.)

10. Find a matrix for each of the given linear transformations of R^3.

(a) Reflection in the yz-plane
(b) Projection onto the yz plane
(c) Reflection through the origin
(d) Reflection through the x-axis

11. Consider a dust cloud in which each particle is moving in uniform circular motion counterclockwise about the z-axis with constant angular velocity ω. For a particle that has initial position (x, y, z) and that moves in this way for T time units, find the new position. Show that this new position is a linear function of (x, y, z)