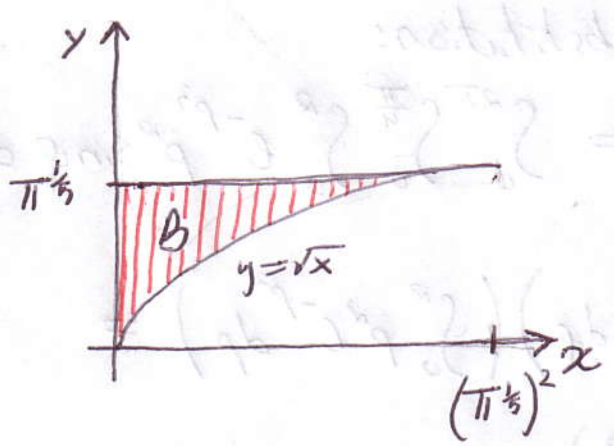
Solutions to Exam 3

1. $\iiint_B xyz = \int_0^1\int_0^1\int_0^1 xyz\,dz\,dy\,dx = \left(\int_0^1 x\,dx\right)^3 =$

$= \frac{1}{8}.$

2. The integral is easily solved by reversing the order of integration. The region of integration is displayed below:



Therefore,

$$\int_0^{(\pi^{1/5})^2}\int_{\sqrt{x}}^{\pi^{1/5}} x\sin(y^5)\,dy\,dx = \int_0^{\pi^{1/5}}\int_0^{y^2} x\sin(y^5)\,dx\,dy$$

$$= \int_0^{\pi^{1/5}} \frac{x^2}{2}\sin(y^5)\Big|_{x=0}^{x=y^2} dy = \int_0^{\pi^{1/5}} \frac{y^4}{2}\sin(y^5)\,dy =$$

$$= \frac{1}{10}\int_0^{\pi^{1/5}} 5y^4\sin(y^5)\,dy = \frac{1}{10}\cdot 2 = \frac{1}{5}.$$

3. a) The surface $z=\sqrt{x^2+y^2}$ is a cone generated by rotating the line $z=x$ about the z-axis.
The surface $x^2+y^2+z^2=4$ is a sphere of radius 2.

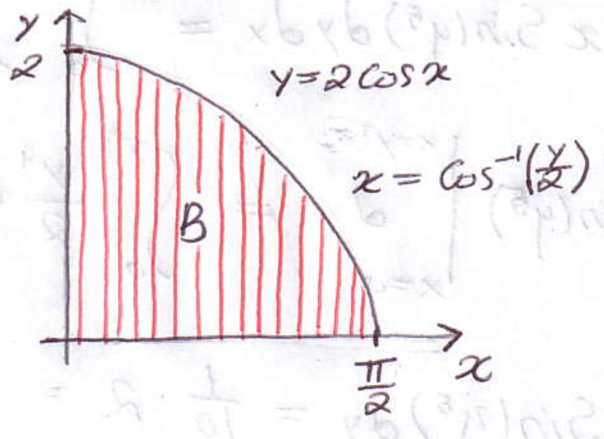
b) The cone and the sphere are easily described in cylindrical as well as spherical coordinates. Since the power to which e is raised, $-(x^2+y^2+z^2)^{3/2}$ is easily described with spherical coordinates, we evaluate the integral, using spherical coordinate substitution:

$$\iiint_B e^{-(x^2+y^2+z^2)^{3/2}} = \int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_0^2 e^{-\rho^3}\rho^2\sin\varphi\, d\rho\, d\varphi\, d\theta =$$

$$= \left(\int_0^{2\pi} d\theta\right)\left(\int_0^{\frac{\pi}{4}}\sin\varphi\, d\varphi\right)\left(\int_0^2 \rho^2 e^{-\rho^3} d\rho\right) =$$

$$= 2\pi\left(1-\frac{1}{\sqrt{2}}\right)\left(\frac{1}{3}-\frac{1}{3}e^{-8}\right).$$

4. a) The region of integration is y-simple and x-simple.



b) $\int_0^{\frac{\pi}{2}}\int_0^{2\cos(x)} f(x,y)\,dy\,dx = \int_0^2\int_0^{\cos^{-1}(\frac{y}{2})} f(x,y)\,dx\,dy$

5. Let $s = y - x^2$ and $t = 3x - y$, then

$$\det \frac{\partial(s,t)}{\partial(x,y)} = \begin{vmatrix} -2x & 1 \\ 3 & -1 \end{vmatrix} = 2x - 3$$

Since $x \le 0$, $2x-3 < 0$ so $\left|\det \frac{\partial(s,t)}{\partial(x,y)}\right| = -(2x-3)$

and $\left|\det \frac{\partial(x,y)}{\partial(s,t)}\right| = \frac{-1}{2x-3}$.

Thus,

$$\iint_B \pi(2x-3)\cos(\pi x^2 - \pi y) = \int_{-4}^{0}\int_0^1 -\pi(2x-3)\cos(\pi(x^2-y))\frac{1}{2x-3}\,dt\,ds$$

$$= \int_{-4}^{0}\int_0^1 -\pi\cos(-\pi s)\,dt\,ds = \int_{-4}^{0} -\pi\cos(\pi s)\,ds = 0.$$

6. Linear maps $T: \mathbb{R}^3 \to \mathbb{R}^3$ change the volume of any solid by the factor $|\det M(T)|$. Now $V(T(S)) =$

$$= V(S)\,|\det M(T)| = (\pi r^2)(2\pi R)\left|\det\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 2 & -1 \end{pmatrix}\right| =$$

$$= (\pi r^2)(2\pi R)\left|\det\begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}\right| = (\pi r^2)(2\pi R)\cdot 3$$

7. Problem 7 is easily solved if we change to polar coordinates. You should verify that region B is described in polar coordinates as

$$0 \leq \theta \leq \pi$$
$$0 \leq r \leq \sin\theta .$$

Therefore,

$$\iint_B \frac{-1}{\sqrt{1-x^2-y^2}} = \int_0^{\pi}\int_0^{\sin\theta} \frac{-r}{\sqrt{1-r^2}}\,dr\,d\theta = \int_0^{\pi}\left(\sqrt{1-\sin^2\theta}-1\right)d\theta =$$

$$= \int_0^{\pi}\left(|\cos\theta|-1\right)d\theta = 2\int_0^{\frac{\pi}{2}}\cos\theta\,d\theta - \pi = 2-\pi .$$

8. The region B is z-simple:

$$0 \leq z \leq x^2+y^2$$
$$(x,y) \in \pi_{(x,y)}(B)$$

where $\pi_{(x,y)}(x,y,z) = (x,y,0)$. Observe that $\pi_{(x,y)}(B)$ is the quarter-elliptical disc $x^2+y^2/9=1$; $x \geq 0$, $y \geq 0$.

Thus,

$$\iiint_B f = \int_0^1\int_0^{3\sqrt{1-x^2}}\int_0^{x^2+y^2} f(x,y,z)\,dz\,dy\,dx \text{ or equivalently}$$

$$\iiint_B f = \int_0^3\int_0^{\sqrt{1-\frac{y^2}{9}}}\int_0^{x^2+y^2} f(x,y,z)\,dz\,dx\,dy .$$

9. Solids of revolution are easily described in cylindrical coordinates. $z = x^2+y^2$ becomes $w = r^2$ and $z = 8-x^2-y^2$ becomes $w = 8-r^2$. The bounds on r can be obtained by setting $r^2 = 8-r^2$ (why?). Thus, $r^2 = 4$ or $r = 2$ is r's upper bound.

The region B can be described as

$$0 \le \theta \le 2\pi$$
$$0 \le r \le 2$$
$$r^2 \le w \le 8-r^2$$

Therefore,

$$\iiint_B \left(1+\frac{x}{\sqrt{x^2+y^2}}\right) = \int_0^{2\pi}\int_0^2\int_{r^2}^{8-r^2} (r + r\cos\theta)\,dw\,dr\,d\theta =$$

$$= \int_0^{2\pi}\int_0^2 (8-2r^2)r(1+\cos\theta)\,dr\,d\theta = \left(\int_0^{2\pi}(1+\cos\theta)d\theta\right)\cdot$$

$$\left(\int_0^2 (8r-2r^3)dr\right) = 2\pi\left(4r^2-\frac{1}{2}r^4\right)\Big|_0^2 = (16-8)2\pi = 16\pi$$

Remark: Notice that $\frac{x}{\sqrt{x^2+y^2}}$ is odd with respect to the yz-plane. Since B is symmetric through the yz-plane

$$\iiint_B \frac{x}{\sqrt{x^2+y^2}} = 0. \text{ Thus } \iiint_B \left(1+\frac{x}{\sqrt{x^2+y^2}}\right) = \iiint_B 1 =$$

$= Vol(B)$.

10. Let $I = \int_{-\infty}^{\infty} e^{-2x} dx$.

Then,

$$I^2 = I \cdot I = \left(\int_{-\infty}^{\infty} e^{-2x} dx \right) \left(\int_{-\infty}^{\infty} e^{-2y} dy \right) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-2(x^2+y^2)} dx dy =$$

$$= \int_0^{2\pi} \int_0^{\infty} r e^{-2r^2} dr d\theta = (-2\pi) \frac{1}{4} \int_0^{\infty} -4r e^{-2r^2} dr =$$

$$= -\frac{\pi}{2} \left(e^{-2(\infty)^2} - e^0 \right) = \frac{-\pi}{2}(-1) = \frac{\pi}{2}$$

Thus

$$I = \sqrt{I^2} = \sqrt{\frac{\pi}{2}}.$$

11. The product of the first two numbers is xy.
This product is bigger than the third number z, whenever $0 \le z \le xy$.
The desired probability is

$$\int_0^1 \int_0^1 \int_0^{xy} dz dy dx = \int_0^1 \int_0^1 xy \, dy dx = \left(\int_0^1 x \, dx \right)^2 = \frac{1}{4}.$$

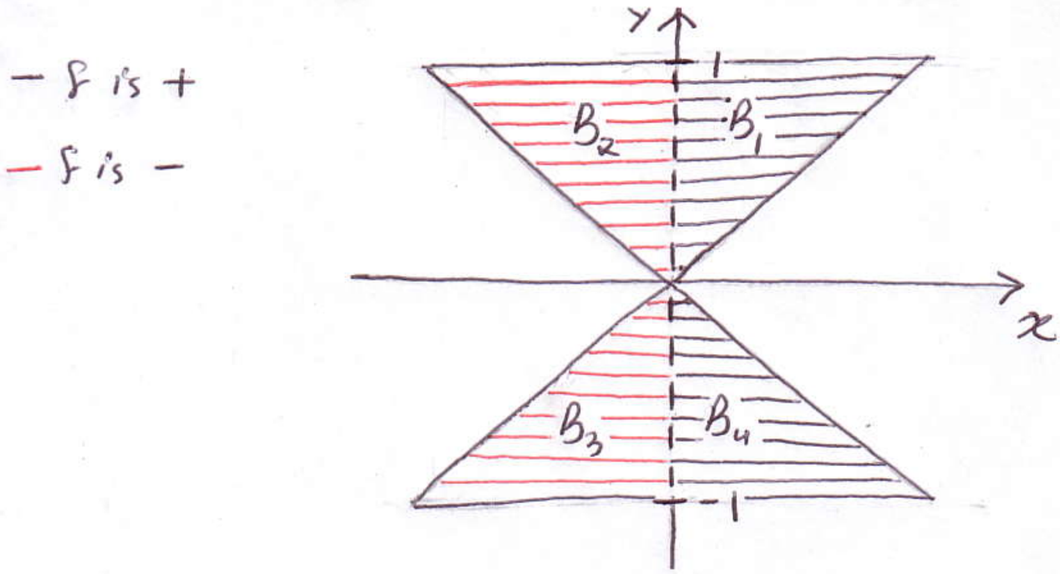
Thus, in one out of four trials we expect the product of the first two numbers to be bigger than the third.

12. Observe that $\sin^2(\ln(1+(xy)^2)) \ge 0$. Therefore

$f(x,y) = 2x \sin^2(\ln(1+(xy)^2))$ satisfies

$f(-x,y) = -f(x,y)$.

Notice that region B is symmetric about the y-axis:



Thus,

$$\iint_B f(x,y) = \iint_{B_1} f(x,y) + \iint_{B_2} f(x,y) + \iint_{B_3} f(x,y) +$$

$$+ \iint_{B_4} f(x,y) = \iint_{B_1} f - \left(-\iint_{B_2} f(x,y)\right) + \iint_{B_3} f - \left(-\iint_{B_4} f(x,y)\right) =$$

$$= \iint_{B_1} f - \iint_{B_2} f(-x,y) + \iint_{B_3} f - \iint_{B_4} f(-x,y) =$$

$$= \iint_{B_1} f - \iint_{B_1} f + \iint_{B_3} f - \iint_{B_3} f = 0$$