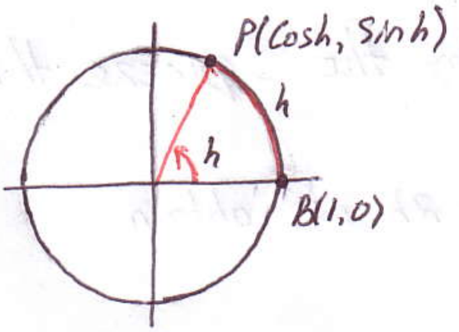
Trigonometric Limits

Thm: $\lim_{h \to 0} \frac{\cos h - 1}{h} = 0$

Proof:



Construct the angle h in standard position. Since the terminal side of h intersects the unit circle at the point P(Cosh, Sinh) and since h is measured in radians, the signed arc length along the unit circle from B(1,0) to P(Cosh, Sinh) is h. Thus, the arc length along the circle between these points, we have

$$0 \leq \sqrt{(1-\cos h)^2 + (0 - \sin h)^2} \leq |h|$$

On squaring and simplifying we get

$$0 \leq 2 - 2\cos h \leq h^2. \qquad (1)$$

If h is positive, then on division by 2h, (1) becomes

$$0 \leq \frac{1-\cos h}{h} \leq \frac{1}{2} h$$

But

$$\lim_{h\to 0^+} \tfrac{1}{2}h = 0 \quad \text{and} \quad \lim_{h\to 0^+} 0 = 0$$

so that

$$\lim_{h\to 0^+} \frac{1-\cos h}{h} = 0$$ by the Squeeze thm. If h is negative,

then on dividing (1) by $2h$ we obtain

$$\tfrac{1}{2}h \leq \frac{1-\cos h}{h} \leq 0$$

But

$$\lim_{h\to 0^-} \tfrac{1}{2}h = 0 \quad \text{and} \quad \lim_{h\to 0^-} 0 = 0$$

So that

$$\lim_{h\to 0^-} \frac{1-\cos h}{h} = 0$$ by the Squeeze thm.

We conclude that $\lim_{h\to 0} \frac{1-\cos h}{h} = 0$ from which it

follows that

$$\lim_{h\to 0} \frac{\cos h - 1}{h} = \lim_{h\to 0} \left(-\left(\frac{1-\cos h}{h}\right)\right) = 0.$$

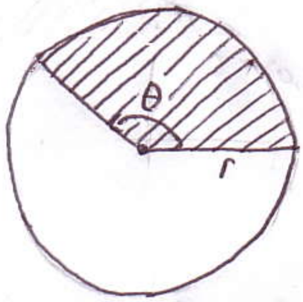
Corollary: $\lim_{h\to 0} \cos h = 1$

Proof:

$$\lim_{h\to 0} \cos h = \lim_{h\to 0}\left(1 + \frac{\cos h - 1}{h}\cdot h\right) = \lim_{h\to 0} 1 + \left(\lim_{h\to 0}\frac{\cos h - 1}{h}\cdot \lim_{h\to 0} h\right)$$

$$= 1 + (0\cdot 0) = 1$$

Before proceeding to the last theorem we must develop the formula for the area of a circular sector. Consider a sector with radius r and a central angle of θ radians



If $\theta = 2\pi$, the sector is the full circle that has area πr^2. Thus, if we assume that the area A of an arbitrary sector is proportional to its central angle θ, we obtain

$$\frac{\text{area of sector}}{\text{area of circle}} = \frac{\text{central angle of sector}}{\text{central angle of circle}}$$

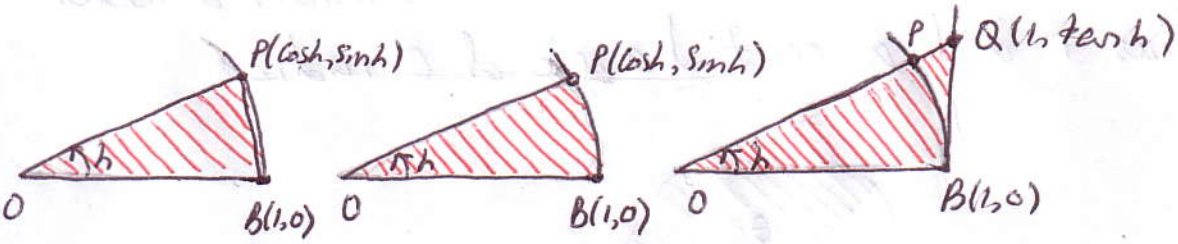
or $$\frac{A}{\pi r^2} = \frac{\theta}{2\pi}$$

Solving for A we obtain the following formula

$$A = \frac{1}{2} r^2 \theta .$$

Thm: $$\lim_{h \to 0} \frac{\sin h}{h} = 1$$

Proof: Assume that h satisfies $0 < h < \pi/2$ and construct the angle h in standard position. The terminal side of h intersects the unit circle at $P(\cos h, \sin h)$ and intersects the vertical line through $B(1,0)$ at the point $Q(1, \tan h)$



By the above figure

$$0 < \text{area of } \Delta OBP < \text{area of sector } OBP < \text{area of } \Delta OBQ$$

But

$\text{area } \Delta OBP = \frac{1}{2} \cdot \text{base} \cdot \text{altitude} = \frac{1}{2} \cdot 1 \cdot \sin h = \frac{1}{2} \sin h$

$\text{area of sector } OBP = \frac{1}{2}(1)^2 \cdot h = \frac{1}{2} h$

$\text{area } \Delta OBQ = \frac{1}{2} \cdot \text{base} \cdot \text{altitude} = \frac{1}{2} \cdot 1 \cdot \tan h = \frac{1}{2} \tan h$

Therefore,

$$0 < \frac{1}{2} \sin h < \frac{1}{2} h < \frac{1}{2} \tan h$$

Multiplying through by $\frac{2}{\sin h}$ yields

$$\cos h < \frac{\sin h}{h} < 1$$

By the above corollary, we know that $\lim_{h \to 0} \cos h = 1$.

Also, $\lim_{h \to 0} 1 = 1$. Hence, by the Squeeze thm, $\lim_{h \to 0} \frac{\sin h}{h} = 1$.