The Substitution Rule Lecture 7

whenever you think about a topic, ask yourself: what would the Native Americans have named it? what is the spirit name of the subject at hand? The substitution rule or u-substitution is simply a mechanism to deal with the problem of reversing chain rule.

Ex. Place a mathematical expression in the box (other than 0) to make integration as simple as possible.

(a) $\int e^{\sin x} \square \, dx$ (b) $\int \sec x^3 \tan x^3 \square \, dx$

(c) $\int \ln x \square \, dx$ (d) $\int \frac{1}{v^3 - 2v^2 + v} \square \, dv$

(e) $\int_1^2 \cos(\pi x^2) \square \, dx$ (f) $\int_1^{27} \frac{1}{x^{2/3} + 1} \square \, dx$

Solution: Feeling clever, you might observe that placing the expression $\frac{1}{f(x)}$ into the box of $\int f(x) \square \, dx$

will result in $\int 1\,dx = x + C$.

Aside from these ideas, we may try to make the integrand look like it came from Chain Rule.

(a) $$\int e^{\sin x} \boxed{\cos x}\, dx = e^{\sin x} + C.$$

(b) $$\int \sec x^3 \tan x^3 \boxed{3x^2}\, dx = \sec x^3 + C$$

(c) $$\int \ln x \boxed{\frac{1}{x}}\, dx = \frac{1}{2}(\ln x)^2 + C$$

(d) $$\int \frac{1}{v^3 - 2v^2 + v} \boxed{3v^2 - 4v + 1}\, dv = \ln\left|v^3 - 2v^2 + v\right| + C$$

(e) $$\int_1^2 \cos(\pi x^2) \boxed{2\pi x}\, dx = \sin(\pi x^2)\Big|_1^2 = 0$$

(f) $$\int_1^{27} \frac{1}{(x^{\frac{1}{3}})^2 + 1} \boxed{\frac{1}{3} x^{-\frac{2}{3}}}\, dx =$$

$$= \tan^{-1}(x^{\frac{1}{3}})\Big|_1^{27} = \tan^{-1} 3 - \tan^{-1} 1$$

$$= \tan^{-1} 3 - \frac{\pi}{4}.$$

Observe that in undoing chain rule, we only need to find the antiderivative of the outermost function.

For example

$$\int 2x\sqrt{1+x^2}\,dx = \int \sqrt{\boxed{1+x^2}}\;\boxed{2x}'\,dx$$

inner function   derivative of inner function

undo $\int \sqrt{\ }$ $= \frac{2}{3}\Box^{3/2}$

$$= \frac{2}{3}(1+x^2)^{3/2} + C$$

or $$\int \sin(\ln x)\,\frac{1}{x}\,dx = \int \sin\left(\boxed{\ln x}\right)\boxed{\frac{1}{x}}'\,dx$$

undo $\int \sin = -\cos$

or $-\cos(\ln x) + C.$

In general, the chain rule pattern looks like

$$\int f'(g(x))\, g'(x)\, dx.$$

The substitution rule emphasizes that only the antiderivative of f' must be found:

$$u = g(x) \qquad \frac{du}{dx} = g'(x) \qquad \text{or} \qquad du = g'(x)\,dx$$

identifies inner function

derivative of inner function.

Then $\displaystyle\int f'(g(x))\, g'(x)\, dx = \int f'(u)\, du$

The last expression makes it plain that we only need to deal with f'.

Ex. Evaluate $\displaystyle\int \sqrt{2x+1}\, dx$

Solution:

(i) Without u-sub

$$\frac{1}{2}\int \sqrt{\boxed{2x+1}}\; \boxed{2}'\, dx = \frac{1}{2}\cdot\frac{2}{3}(2x+1)^{3/2} + C = \frac{1}{3}(2x+1)^{3/2} + C.$$

(ii) with u-sub

$$\int \sqrt{2x+1}\, dx \qquad \text{set } u = 2x+1, \; du = 2dx$$

$$\tfrac{1}{2} du = dx$$

$$\int \sqrt{u}\, \tfrac{1}{2} du = \tfrac{1}{2} \underbrace{\int \sqrt{u}\, du}_{\text{integrate outer function.}} = \tfrac{1}{3} u^{3/2} + C$$

$$= \tfrac{1}{3} (2x+1)^{3/2} + C.$$

Ex. Find $\displaystyle\int \frac{x}{\sqrt{1-4x^2}}\, dx$

Solution:

(i) without u-sub

$$-\frac{1}{8} \int \frac{1}{\sqrt{\boxed{1-4x^2}}} \; \boxed{-8\,x}'\, dx = -\frac{1}{4}\sqrt{1-4x^2} + C$$

(ii) with u-sub

$$u = 1-4x^2 \qquad du = -8x\, dx, \quad -\tfrac{1}{8} du = x\, dx$$

$$\int \frac{1}{\sqrt{u}} \left(-\tfrac{1}{8}\right) du = 2\sqrt{u}\left(-\tfrac{1}{8}\right) + C = -\tfrac{1}{4}\sqrt{1-4x^2} + C.$$

Ex. Find $\int e^{5x}\,dx$

Solution:

$$\frac{1}{5}\int e^{\boxed{5x}}\,\boxed{5}'\,dx = \frac{1}{5}e^{5x}+C.$$

Ex. Find $\int x^3\cos(x^4+2)\,dx$

Solution:

$$\frac{1}{4}\int \boxed{4x^3}'\cos\left(\boxed{x^4+2}\right)dx = \frac{1}{4}\sin(x^4+2)+C.$$

Ex. Calculate $\int \tan x\,dx$

Solution:

$$\int\frac{\sin x\,dx}{\cos x} = -1\int\frac{1}{\boxed{\cos x}}\,\boxed{-\sin x}'\,dx$$

$$= -\ln|\cos x|+C = \ln|\sec x|+C.$$

You are convinced, I hope, that in most instances it is simpler to guess the antiderivative without going through formalisms of u-substitution. There are some instances when u-sub is useful, Ex. Find $\int \sqrt{1+x^2}\; x^5\, dx$.

Solution:

This integral does not seem to involve reverse Chain Rule. Notice however that $\sqrt{\cdot}$ makes the whole thing difficult. It would be nice to get it removed.

Set $u = \sqrt{1+x^2}$, then $u^2 = 1+x^2$ and $2u\,du = 2x\,dx$

or $u\,du = x\,dx$

Now $x^5 dx = x^4 x\,dx = (x^2)^2 x\,dx = (u^2-1)^2 u\,du$

So $\int \sqrt{1+x^2}\; x^5 dx = \int u\,(u^2-1)^2 u\,du =$

$= \int (u^2-1)^2 u^2\,du = \int (u^6 - 2u^4 - u^2)\,du$

$= \frac{1}{7}u^7 + \frac{-2}{5}u^5 - \frac{1}{3}u^3 + C =$

$= \frac{1}{7}(1+x^2)^{7/2} - \frac{2}{5}(1+x^2)^{5/2} - \frac{1}{3}(1+x^2)^{3/2} + C.$

when dealing with definite integrals, observe that

$$\int_a^b f'(g(x))\,g'(x)\,dx = f(g(x))\Big|_a^b = f(g(b)) - f(g(a))$$

$$= f(u)\Big|_{u=g(a)}^{u=g(b)} = \int_{g(a)}^{g(b)} f'(u)\,du.$$

Ex. $\displaystyle\int_0^4 \sqrt{2x+1}\,dx = \frac{1}{2}\int_{0+1}^{8+1} \sqrt{u}\,du$

$$= \frac{1}{2}\int_1^9 \sqrt{u}\,du = \frac{1}{2}\cdot\frac{2}{3}u^{3/2}\Big|_1^9 = \frac{1}{3}(27-1) = \frac{26}{3}.$$

Ex. Evaluate $\displaystyle\int_1^2 \frac{dx}{(3-5x)^2}$

Solution:

$u = 3-5x$; $du = -5\,dx \Rightarrow dx = -\frac{1}{5}du$

$u(1) = 3-5 = -2$, $u(2) = 3-5\cdot 2 = -7$

$$-\frac{1}{5}\int_{-2}^{-7} \frac{du}{u^2} = -\frac{1}{5}\left[-\frac{1}{u}\right]_{-2}^{-7} = -\frac{1}{5}\left(\frac{1}{7}-\frac{1}{2}\right) = \frac{1}{14}.$$

Ex. Calculate $\int_1^e \frac{\ln x}{x}\,dx$

Solution: $u = \ln x$, $du = \frac{1}{x}$

$$\int_{\ln 1}^{\ln e} u\,du = \int_0^1 u\,du = \left.\frac{u^2}{2}\right|_0^1 = \frac{1}{2}.$$

Ex. Find

(a) $\int \frac{(\ln x)^2}{x}\,dx$

(b) $\int \sec^2\theta \tan^3\theta\,d\theta$

(c) $\int \sqrt{x}\,\sin(1+x^{3/2})\,dx$

(d) $\int \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx$

(e) $\int 5^t \sin(5^t)\,dt$

(f) $\int \frac{1+x}{1+x^2}\,dx$

(g) $\int x^2\sqrt{2+x}\,dx$

(h) $\int x^3\sqrt{x^2+1}\,dx$

Solution:

(a) $\int \frac{(\ln x)^2}{x}\,dx = \frac{1}{3}(\ln x)^3 + C$

(b) $\int \sec^2\theta \tan^3\theta \, d\theta = \frac{1}{4}\tan^4\theta + C.$

(c) $\frac{2}{3}\int \frac{3}{2}\sqrt{x}\,\sin(1+x^{3/2})\,dx = -\frac{2}{3}\cos(1+x^{3/2}) + C.$

(d) $2\int \frac{\sin\sqrt{x}}{2\sqrt{x}}\,dx = -\cos\sqrt{x} + C.$

(e) $\frac{1}{\ln 5}\int 5^t \ln 5 \,\sin(5^t)\,dt = -\frac{1}{\ln 5}\cos(5^t) + C.$

(f) $\int \frac{1+x}{1+x^2}\,dx = \int \frac{1}{1+x^2}\,dx + \frac{1}{2}\int \frac{2x}{1+x^2}\,dx$

$= \tan^{-1}x + \frac{1}{2}\ln(1+x^2) + C.$

(g) set $u = \sqrt{2+x}$ then $u^2 = 2+x$,

$x = u^2 - 2$, $dx = 2u\,du$

so $\int x^2\sqrt{2+x}\,dx = \int (u^2-2)^2 u\cdot 2u\,du =$

$= \int (u^2-2)^2\cdot 2u^2\,du = 2\int (u^4 - 4u^2 + 4)u^2\,du$

$= 2\int (u^6 - 4u^4 + 4u^2)\,du = 2\left(\frac{u^7}{7} - \frac{4}{5}u^5 + \frac{4}{3}u^3\right) + C$

$= \frac{2}{7}\left(\sqrt{2+x}\right)^7 - \frac{8}{5}\left(\sqrt{2+x}\right)^5 + \frac{8}{3}\left(\sqrt{2+x}\right)^3 + C.$

(h) $\int x^3 \sqrt{x^2+1}\, dx$

$u = \sqrt{x^2+1}$; $u^2 = x^2+1$; $u\,du = x\,dx$

$$\int (u^2-1)\, u^2\, du = \int (u^4 - u^2)\, du = \frac{1}{5}u^5 - \frac{1}{3}u^3 + c$$

$$= \frac{1}{5}\left(x^2+1\right)^{5/2} - \frac{1}{3}\left(x^2+1\right)^{3/2} + C.$$

<u>Ex.</u> Solve

(a) $\displaystyle\int_1^2 \frac{e^{\frac{1}{x}}}{x^2}\, dx$ (b) $\displaystyle\int_1^2 x\sqrt{x-1}\, dx$

(c) $\displaystyle\int_e^{e^4} \frac{dx}{x\sqrt{\ln x}}$ (d) $\displaystyle\int_0^1 \frac{e^z+1}{e^z+z}\, dz$

(e) $\displaystyle\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} x^4 \sin x\, dx$ (f) $\displaystyle\int_0^{\frac{1}{2}} \frac{\sin^{-1}x}{\sqrt{1-x^2}}\, dx$

(g) $\displaystyle\int_0^{\frac{\pi}{2}} \cos x \sin(\sin x)\, dx$ (h) $\displaystyle\int_0^1 \frac{dx}{(1+\sqrt{x})^4}$

Solution:

(a) $u = \frac{1}{x}$ $du = -\frac{1}{x^2}dx$

$$-\int_1^{\frac{1}{2}} e^u\,du = \int_{\frac{1}{2}}^1 e^u\,du = e^1 - e^{\frac{1}{2}} = e - \sqrt{e}.$$

(b) $u = x-1$, $du = dx$

$$\int_0^1 (u+1)\sqrt{u}\,du = \int_0^1 \left(u^{\frac{3}{2}} + u^{\frac{1}{2}}\right) du$$

$$= \frac{2}{5}u^{\frac{5}{2}} + \frac{2}{3}u^{\frac{3}{2}}\Big|_0^1 = \frac{2}{5} + \frac{2}{3}$$

(c) $u = \ln x$, $du = \frac{1}{x}$

$$\int_1^4 \frac{1}{\sqrt{u}}\,du = 2\sqrt{u}\Big|_1^4 = 2(2-1) = 2.$$

(d) $u = e^z + z$ $du = (e^z + 1)dz.$

$$\int_1^{e+1} \frac{1}{u}\,du = \ln|u|\Big|_1^{e+1} = \ln(e+1)$$

(e) $x^4 \sin x$ is an odd function $\Rightarrow \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} x^4 \sin x\,dx = 0.$

(f) $\displaystyle\int_0^{\frac{1}{2}} \frac{\sin^{-1} x}{\sqrt{1-x^2}}\, dx = \frac{1}{2}\left[\sin^{-1}(x)\right]^2 \Big|_0^{\frac{1}{2}}$

$= \frac{1}{2} \cdot \frac{\pi}{6} = \frac{\pi}{12}.$

(g) $\displaystyle\int_0^{\frac{\pi}{2}} \cos x \sin(\sin x)\, dx = \int_0^1 \sin(u)\, du =$

$= -\cos(u)\Big|_0^1 = 1 - \cos(1).$

(h) set $u = 1+\sqrt{x}$, then $(u-1)^2 = x$ and

$2(u-1)\, du = dx$

$$\int_1^2 \frac{2(u-1)}{u^4}\, du = 2\int_1^2 (u^{-3} - u^{-4})\, du =$$

$$= 2\left[-\frac{1}{2}u^{-2} + \frac{1}{3}u^{-3}\right]_1^2 = \frac{1}{6}.$$

Symmetry

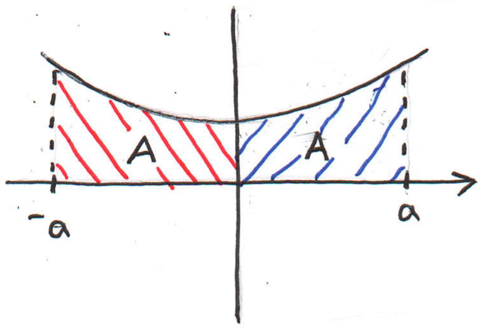
suppose f is continuous on $[-a, a]$

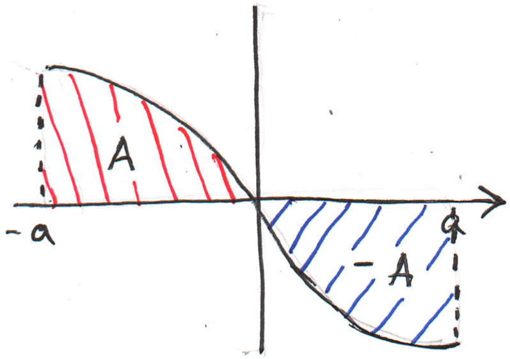
(i) If f is even (i.e. $f(-x) = f(x)$) then

$$\int_{-a}^{a} f(x)\, dx = 2\int_0^a f(x)\, dx$$

(ii) If f is odd (i.e. $f(-x) = -f(x)$) then

$$\int_{-a}^{a} f(x)\,dx = 0$$





f even:

$$\int_{-a}^{a} f(x)\,dx = 2A = 2\int_{0}^{a} f(x)\,dx.$$

f odd:

$$\int_{-a}^{a} f(x)\,dx = A - A = 0.$$

Ex. $$\int_{-2}^{2} (x^2+5)\,dx = 2\int_{0}^{2} (x^2+5)\,dx =$$

$$= 2\left[\tfrac{1}{3}x^3 + 5x\right]_0^2 = 2\left(\frac{8}{3} + 10\right) = \frac{16}{3} + 20.$$

Ex. $$\int_{-1}^{1} \frac{\tan x}{1+x^2+x^3}\,dx = 0$$ because the integrand is an odd function.