Area under curve Lecture 2

In this section we will introduce the second main problem of calculus. Before we do that, it will be helpful to introduce a new notation for sum.

Sigma Notation

$$\sum_{k=1}^{n} a_k = a_1 + a_2 + a_3 + \ldots + a_{n-1} + a_n$$

This notation tells you how to consecutively sum terms that begin with the index at bottom and end with index on top of sigma.

Ex. Write in expended form.

(a) $\displaystyle\sum_{k=1}^{5} \frac{1}{k}$ (b) $\displaystyle\sum_{k=3}^{6} 2k$ (c) $\displaystyle\sum_{k=6}^{6} k^3$

(d) $\displaystyle\sum_{j=-2}^{1} j^2$ (e) $\displaystyle\sum_{i=2}^{6} 5$ (f) $\displaystyle\sum_{s=-3}^{0} e^{2s-1}$

Solution:

(a) $$\sum_{k=1}^{5} \frac{1}{k} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5}$$

(b) $\sum_{k=3}^{6} 2k = 2\cdot3 + 2\cdot4 + 2\cdot5 + 2\cdot6 = \underbrace{6 + 8 + 10 + 12}_{\text{sum of even \#}}$

(c) $\sum_{k=6}^{6} k^3 = 6^3$

(d) $\sum_{j=-2}^{1} j^2 = (-2)^2 + (-1)^2 + (0)^2 + (1)^2$

(e) $\sum_{i=2}^{6} 5 = \underset{i=2}{5} + \underset{i=3}{5} + \underset{i=4}{5} + \underset{i=5}{5} + \underset{i=6}{5} = (6-2+1)\cdot5$

(f) $\sum_{s=-3}^{0} e^{2s-1} = e^{-2\cdot3-1} + e^{-2\cdot2-1} + e^{-2\cdot1-1} + e^{-2\cdot0-1}$

$$= \underbrace{e^{-7} + e^{-5} + e^{-3} + e^{-1}}_{\text{powers are negative odd \#.}}$$

Ex. Convert to sigma notation

(a) $2+3+4+5+6+7$

(b) $1+3+5+7+9+11$

(c) $2+4+6+8$

(d) $(-2)+(-2)+(-2)+(-2)$

(e) $\frac{1}{2} + \frac{1}{3} + \frac{1}{4}$

(f) $\frac{1}{4} + \frac{1}{6} + \frac{1}{8}$

(h) $\frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25}$

(i) $3+6+9+12+15+18$

Solution:

(a) $2+3+4+5+6+7 = \sum_{k=2}^{7} k$

(b) $1+3+5+7+9+11 = \sum_{k=0}^{5} (2k+1)$

(c) $2+4+6+8 = \sum_{k=1}^{4} 2k$

(d) $(-2)+(-2)+(-2)+(-2) = \sum_{k=1}^{4} -2$

(e) $\frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \sum_{k=2}^{4} \frac{1}{k}$

(f) $\frac{1}{4} + \frac{1}{6} + \frac{1}{8} = \sum_{k=2}^{4} \frac{1}{2\cdot k}$

(h) $\frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} = \sum_{k=2}^{5} \frac{1}{k^2}$

(i) $3+6+9+12+15+18 = \sum_{k=1}^{6} 3k$

Remark: The letter k in the examples above is called dummy variable. We can exchange it to any other letter

e.g. $\sum_{k=1}^{6} 3k = \sum_{i=1}^{6} 3i$

Always read the meaning! Different looking sums might mean the same thing. For example

$$\sum_{k=1}^{6} k^2 = \sum_{k=3}^{8} (k-2)^2$$

Both sums represent consecutive sums of squares $1^2+2^2+3^2+4^2+5^2+6^2$.

The following properties hold

1. $$\sum_{k=m}^{n} (a_k+b_k) = \sum_{k=m}^{n} a_k + \sum_{k=m}^{n} b_k$$

2. $$\sum_{k=m}^{n} c\,a_k = c\sum_{k=m}^{n} a_k .$$

To see this, observe that

1. $$\sum_{k=m}^{n} (a_k+b_k) = (a_m+b_m)+(a_{m+1}+b_{m+1})+\ldots+(a_n+b_n)$$

$$= (a_m+a_{m+1}+\ldots+a_n)+(b_m+b_{m+1}+\ldots+b_n)$$

$$= \sum_{k=m}^{n} a_k + \sum_{k=m}^{n} b_k$$

2. $$\sum_{k=m}^{n} c\,a_k = c\,a_m + c\,a_{m+1} + \ldots + c\,a_n$$

$$= c(a_m+a_{m+1}+\ldots+a_n) = c\sum_{k=m}^{n} a_k .$$

Some important sums

1. $$\sum_{k=m}^{n} a = \underset{\substack{\downarrow \\ m}}{a} + \underset{\substack{\downarrow \\ m+1}}{a} + \dots + \underset{\substack{\downarrow \\ n}}{a}$$

is just the constant a added to itself a bunch of times? How many times? Look again

$$\underset{\substack{\downarrow \\ m+0}}{a} + \underset{\substack{\downarrow \\ m+1}}{a} + \underset{\substack{\downarrow \\ m+2}}{a} + \dots + \underset{\substack{\downarrow \\ m+(n-m)}}{a}$$

The count is from 0 to $n-m$. Hence a is added to itself $(n-m+1)$ times. Thus

$$\sum_{k=m}^{n} a = (n-m+1)\, a.$$

2. $$\sum_{k=1}^{n} k = 1+2+3+\dots+(n-2)+(n-1)+n$$

Legend has it that a cruel German elementary school teacher was making his pupils calculate sums of the type $1+2+3+\dots+100$.
The poor pupils would toil for hours, inevitably getting the wrong answer. All but one, later to be known as the prince of mathematics, Carl Friedrich Gauss.

At the age of 6 he would instantly write the answer on the 19th century tablet (which was just that!) exclaiming "Hier steht's!".
How might he have done that?

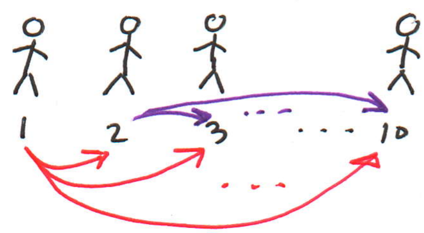
$$
\begin{aligned}
S &= 1 + 2 + 3 + \dots + (n-2) + (n-1) + n \\
+\quad S &= n + (n-1) + (n-2) + \dots + 3 + 2 + 1 \\
\hline
2S &= \underset{1}{(n+1)} + \underset{2}{(n+1)} + \underset{3}{(n+1)} + \dots + \underset{n-2}{(n+1)} + \underset{n-1}{(n+1)} + \underset{n}{(n+1)}
\end{aligned}
$$

Hence $2S = n(n+1) \implies S = \dfrac{n(n+1)}{2}$

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

Ex. 10 diplomats are in a meeting. How many handshakes take place if every diplomat must shake hands once with every other diplomat?

Solution:

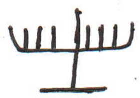


9 handshakes
+8 handshakes
⋮
+1 handshake

$9+8+7+\dots+1$ handshakes take place.

$1+2+\dots+9 = \dfrac{9\cdot 10}{2} = 45.$

<u>Ex.</u> The Menorah has 9 sockets for chanukah candles. The holliday runs for 8 days. On the first day you light 2 candles, on the second day you light 3 candles, ..., on the 8^{th} day – 9 candles. How many candles must be in the candle box?

<u>Solution:</u> 

$$S = 2+3+4+\ldots+9 \qquad S+1 = 1+2+3+4+\ldots+9$$

$$S+1 = \frac{9\cdot 10}{2} = 45 \implies S = 44 \text{ candles.}$$

<u>Ex.</u> Find a closed form for the sum

$$\sum_{k=m}^{n} k$$

<u>Solution:</u>

$$S = m + (m+1) + \ldots + m+(n-m)$$

$$= \underbrace{m}_{0} + 0 + \underbrace{m}_{1} + 1 + \ldots + \underbrace{m}_{n-m} + (n-m) = m(n-m+1) + \sum_{k=1}^{n-m} k$$

$$= m(n-m+1) + \frac{(n-m)(n-m+1)}{2}$$

$$= \frac{2m(n-m+1) + (n-m)(n-m+1)}{2} = \frac{(2m+n-m)(n-m+1)}{2}$$

$$= \frac{(m+n)(n-m+1)}{2}.$$

2. $\displaystyle\sum_{k=1}^{n} k^2 = 1^2 + 2^2 + 3^2 + \dots + n^2.$

This is a little more difficult! If you studied combinatorial analysis, many stories would appear to you in the form of compressed formulas, and conversely, many many formulas would look like short stories.

For example, $\displaystyle\sum_{k=1}^{n} k^2$ looks to me like a man talking about his two brothers. The man is between 2 and $n+1$ years old and has 2 brothers that are younger than him

$\displaystyle\sum_{k=1}^{n} k^2$ counts the possible ages of all 3 siblings:

$$\left|\left(\underset{A}{\boxed{1}},\ \underset{B}{\boxed{1}},\ \underset{C}{2}\right)\right| + \left|\left(\underset{A}{\square},\ \underset{B}{\square},\ \underset{C}{3}\right)\right| + \dots + \left|\left(\underset{A}{\square},\ \underset{B}{\square},\ n+1\right)\right|$$

only one possibility

age of A $\times$ age of B $=1$ or 2 $\quad$ $=1$ or 2

age of A $\times$ age of B $=1,2,\dots,n$ $\quad$ $=1,2,\dots,n$

$$1^2 \quad + \quad 2^2 \quad + \dots + \quad n^2$$

$$= 2\binom{n+1}{3} + \binom{n+1}{2} = 2\,\frac{(n+1)\,n\,(n-1)}{6} + \frac{(n+1)\,n}{2}$$

younger brothers are not twins

younger brothers are twins.

$$= \frac{2(n-1)+3}{6}(n+1)n = \frac{(2n+1)(n+1)n}{6}$$

Since we haven't learned the principles of counting we will derive the solution by means of an algebraic trick. We will need to quickly expand terms of the form $(x+y)^n$:

1

1 1 $\quad (x+y)^1 = 1x + 1y$.

1 2 1 $\quad (x+y)^2 = 1\cdot x^2 + 2xy + 1\cdot y^2$

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

Thus for instance, $(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$

and $(x+y)^6 = x^6 + 6x^5y + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6xy^5 + y^6$.

We are going to compute a seemingly more difficult sum: $\displaystyle\sum_{k=1}^{n}\left[(k+1)^3 - k^3\right]$.

Let's do it in two ways.

(i) $$\sum_{k=1}^{n}\left[(k+1)^3-k^3\right] = \underbrace{\sum_{k=1}^{n}(k+1)^3}_{\text{just sum } 2^3+\dots+(n+1)^3\ !!!} - \sum_{k=1}^{n}k^3$$

$$= \sum_{k=2}^{n+1}k^3 - \sum_{k=1}^{n}k^3 = \boxed{\sum_{k=2}^{n}k^3} + (n+1)^3 - 1^3 - \boxed{\sum_{k=2}^{n}k^3}$$

cancel each other!

$$= (n+1)^3 - 1^3 = n^3+3n^2+3n$$

(ii) On the other hand $(k+1)^3-k^3$

$$= k^3+3k^2+3k+1-k^3 = 3k^2+3k+1$$

Thus $$\sum_{k=1}^{n}\left[(k+1)^3-k^3\right] = \sum_{k=1}^{n}\left(3k^2+3k+1\right)$$

$$= 3\underbrace{\sum_{k=1}^{n}k^2}_{\text{what we wish to find}} + 3\underbrace{\sum_{k=1}^{n}k}_{\frac{n(n+1)}{2}} + \underbrace{\sum_{k=1}^{n}1}_{n} = 3\sum_{k=1}^{n}k^2 + \frac{3n(n+1)}{2} + n$$

$$= 3\sum_{k=1}^{n}k^2 + n\,\frac{3n+3+2}{2}$$

$$= \frac{3n^2+5n}{2} + 3\sum_{k=1}^{n} k^2$$

Thus we have $n^3+3n^2+3n = 3\sum_{k=1}^{n} k^2 + \frac{3n^2+5n}{2}$

$$\sum_{k=1}^{n} k^2 = \frac{n^3+3n^2+3n}{3} - \frac{3n^2+5n}{6}$$

$$= \frac{2n^3+6n^2-3n^2+6n-5n}{6}$$

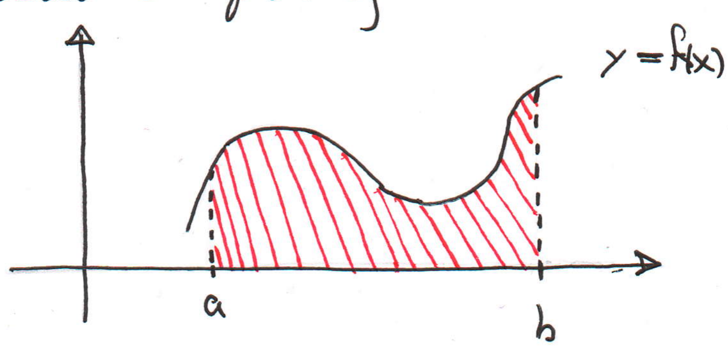
$$= \frac{2n^3+3n^2+n}{6} = \frac{n(2n^2+3n+1)}{6}$$

$$= \frac{n(2n+1)(n+1)}{6}.$$

3. Use the idea in 2. to establish a closed form for $\sum_{k=1}^{n} k^3$.

The area problem

Given a curve $y=f(x)$ over the x-axis, how do we go about computing the area?



In mathematical notation

$$\int_a^b f(x)\,dx = ?$$

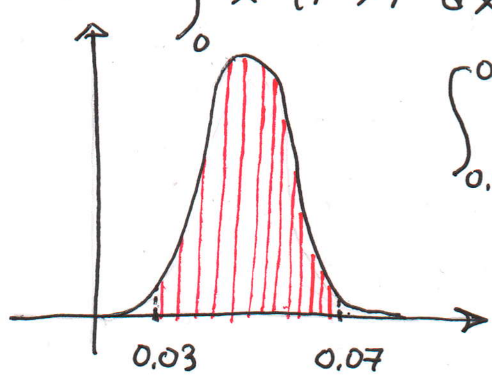
We will see the reason for the notation shortly. First, why would anyone in his right mind obsess about this problem?

Probability

Suppose you wish to measure the prevalance of a certain illness in the general population. You pick a random sample of 1000 individuals and find that 50 have the illness. What can be said about the frequency of this illness in the population?

The problem reduces to calculations of the areas below the curve $f(x) = \frac{1}{B(50,950)} x^{50}(1-x)^{950}$

where $B(50,950) = \int_0^1 x^{50}(1-x)^{950}\,dx$



$$\int_{0.03}^{0.07} f(x)\,dx \approx 0.992$$

In other words, it is more than 99% certain that between 3% to 7% of the population is afflicted. With greater sample size the estimate can be made much more sharp.

The use of integration theory is ubiquitous in probability theory and therefore in search engine designs, machine learning, quantum mechanics, etc

Physics

Fundamental statements about physical quantities such as work, momentum, force, energy are statements about the properties of areas

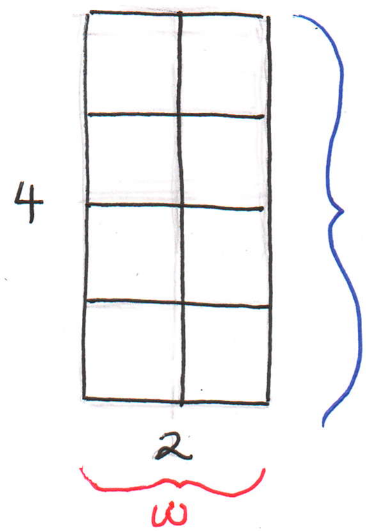
Work is the integral of force over distance, momentum is the integral of force over time.

In fact the statement about conservation of energy is in its simpler form equivalent to the fundamental theorem of calculus. A statement about area under the curve.

The idea

Q. What is the simplest geometric object in the context of area?

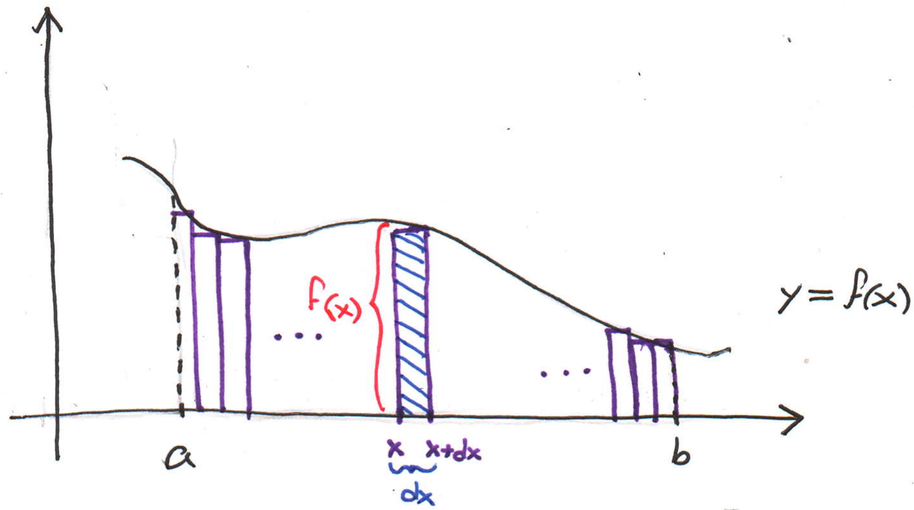
A. A rectangle!



area $= hw$. $(= 2 \cdot 4)$. In its most basic form, the area of a rectangle states that if we have a floor w-units wide and h-units heigh, the number of square unit tiles that cover it $= w \cdot h$.

Rectangles form the "pixels" or building blocks in the calculus description of area, just as lines have been the building blocks for the study of curves.

Given a curve $y = f(x)$ we will stock it with a myriad of tiny rectangles in such a way that their tops fit snuggly along the curve.



The area is then the sum of all the thin rectangles (i.e. their areas) that are stocked between a and b.
The area of the rectangle at x is $f(x)dx$ and the sum of these areas is written as

$$\int_a^b f(x)\,dx$$

where $\int$ stands for S = sum!

In the next lecture we will learn how to apply this idea!