Antiderivatives Lecture 1

we have practiced various techniques of differentiation. Suppose now that, as so often happens you come late to the lecture and you see the solution

$$f'(x) = 2x.$$

What was $f(x)$?

you might predict

$$f(x) = x^2 \quad \text{or} \quad f(x) = x^2 + 5 \quad \text{or} \quad f(x) = x^2 - 7$$

In general, $f(x) = x^2 + c$ is the family of functions for which $f'(x) = 2x$.

Ex. Guess $f(x)$.

(a) $f'(x) = 7$ (b) $f'(x) = 3x^2$ (c) $f'(x) = x^2$

(d) $f'(x) = x^5$ (e) $f'(x) = x^n$ (f) $f'(x) = e^x$.

Solution:

(a) Derivative is constant so the function must be a line! $f(x) = 7x + c$

(b) $f(x) = x^3 + c$

(c) $f(x) = \boxed{\frac{1}{3}} x^3 + C$

(d) $f(x) = \boxed{\frac{1}{6}} x^6 + C$

(e) $f(x) = \boxed{\frac{1}{n+1}} x^{n+1} + C$ Notice that $n \neq -1$.

(f) $f(x) = e^x + C$.

Ex. Find $f(x)$

(a) $f'(x) = e^5$ (b) $f'(x) = \frac{1}{x}$ (c) $f'(x) = x^2 + 5x + e^x$

(d) $f'(x) = \cos x$ (e) $f'(x) = \sin x$ (f) $f'(x) = \sec^2 x$

(g) $f'(x) = \sec x \tan x$ (h) $f'(x) = \cot^2 x$ (i) $f'(x) = \sqrt[3]{x^2} + x\sqrt{x}$

Solution:

(a) $f'(x) = e^5 \cdot x + C$

(b) Recall that $\frac{d}{dx} \ln x = \frac{1}{x}$ when $x > 0$. In general $\frac{d}{dx} \ln|x| = \frac{1}{x}$. Hence we guess $f(x) = \ln x + C$.

(c) Derivative of sum is sum of derivatives and derivatives commute with constants! (i.e. $[c\,k(x)]' = c\,k'(x)$)

$$\underbrace{x^2}_{\downarrow\ \frac{1}{3}x^3} + 5\underbrace{x}_{\downarrow\ \frac{1}{2}x^2} + \underbrace{e^x}_{\downarrow\ e^x} \Longrightarrow f(x) = \frac{1}{3}x^3 + \frac{5}{2}x^2 + e^x.$$

(d) $f(x) = \sin x + C$ (e) $f(x) = -\cos x + C$

(Guess, notice error, modify: $-(\cos x)' \to -(-\sin x)$)

(f) $f(x) = \tan x + C$

(g) $f(x) = \sec x + C$ (h) $f'(x) = \dfrac{\cos^2 x}{\sin^2 x}$ what to do?

Recall trig identities $\cos^2 x + \sin^2 x = 1$.

$$\Longrightarrow \frac{\cos^2 x}{\sin^2 x} + \frac{\sin^2 x}{\sin^2 x} = \frac{1}{\sin^2 x}$$

$$\cot^2 x + 1 = \csc^2 x \quad \text{or}$$

$$\cot^2 x = \csc^2 x - 1$$

Thus $f(x) = -\cot x - x + C$.

(i) $f'(x) = x^{2/3} + x^{3/2} \Longrightarrow f(x) = \boxed{\frac{3}{5}} x^{\overbrace{2/3+1}^{5/3}} + \boxed{\frac{2}{5}} x^{\overbrace{3/2+1}^{5/2}} + C$

<u>Ex.</u> $f'(t) = \dfrac{3t^4 - t^3 + 6t^2}{t^4}$ find $f(t)$.

<u>Solution:</u> $f'(t) = 3 - \frac{1}{t} + 6\frac{1}{t^2} = 3 - \frac{1}{t} + 6t^{-2}$

$f(t) = 3t - \ln|t| - 6t^{-1} + C$.

Ex. Find $f(x)$

(a) $f'(x) = 5x^2 + 2^x + \frac{1}{x}$

(b) $f'(x) = \frac{1}{1+x^2}$

(c) $f'(x) = \frac{1}{\sqrt{1-x^2}}$

(d) $f'(x) = \frac{1}{x\sqrt{x^2-1}}$

Solution:

(a) $$5\underbrace{x^2}_{\downarrow\ \frac{1}{3}x^3} + \underbrace{2^x}_{\downarrow\ 2^x \frac{1}{\ln 2}} + \underbrace{\frac{1}{x}}_{\downarrow\ \ln|x|}$$

$$f(x) = \frac{5}{3}x^3 + 2^x \frac{1}{\ln 2} + \ln|x| + C$$

(b) Recall $\frac{d}{dx} \tan^{-1}x = \frac{1}{1+x^2}$

so $f(x) = \tan^{-1}x + C$

(c) $f(x) = \sin^{-1}x + C$

(d) $f(x) = \sec^{-1}x + C$.

Have you seen the movie "Usual Suspects"?

For the differential equation $f'(x) = 2x$ we have guessed $f(x) = x^2 + C$ as the most general candidate.

Are there possibly more exotic suspects?

For instance $g(x) = \int_0^x 2u\,du$ satisfies $g'(x) = 2x$

Never mind what the formula means! We'll get to that later. $g(x)$ is Keyser Sousse. Is he one of our usual suspects? Is $g(x) = x^2 + c$ for some constant c?

<u>Ex.</u> $f'(x) = \dfrac{1}{1+x^2}$ what is $f(x)$?

<u>Solution:</u> (i) Recall $\dfrac{d}{dx} \tan^{-1} x = \dfrac{1}{1+x^2}$

Guess $f(x) = \tan^{-1} x + C$.

(ii) Recall $\dfrac{d}{dx} \cot^{-1} x = \dfrac{-1}{1+x^2}$

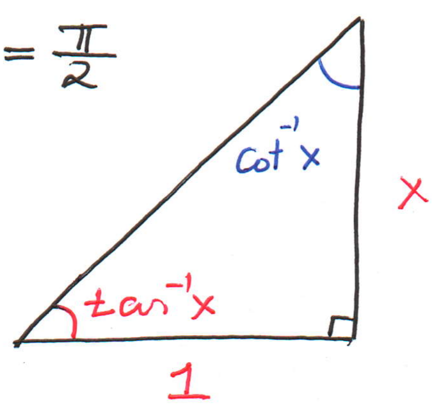
Guess $g(x) = -\cot^{-1} x$.

Is $g(x)$ a novel guess? One that isn't accounted for in the family $\tan^{-1} x + c$?

$\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}$

$g(x) = -\cot^{-1} x$

$= -\frac{\pi}{2} + \tan^{-1} x$



Thus

$-\cot^{-1} x = \tan^{-1} x - \frac{\pi}{2}$

Wow! Keyser Sousse is among the usual suspects!

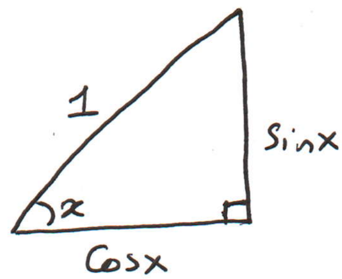
Ex. Let $f(x) = \cos^2 x$ and $g(x) = -\sin^2 x$

then $f'(x) = 2\cos x(-\sin x) = -2\sin x \cos x$

$g'(x) = -2\sin x \cos x$

So $f(x)$ and $g(x)$ have the same derivative

How are these functions related?



$\cos^2 x + \sin^2 x = 1$ so $\cos^2 x - 1 = -\sin^2 x$

or $g(x) = f(x) - 1$.

Q. Suppose $f'(x) = g'(x)$ for two functions $f(x)$ and $g(x)$, what's the best you can say about the relationship between these two?

A. If you asked Freud he might have said: Diese zwei haben eine verborgene Beziehung. In der Öffentlichkeit halten sie sich vonanader in Abstand.
(These two have a hidden relationship. They keep their distance from each other in public)

We can understand this as follows:

Let $k(x) = g(x) - f(x)$. Then $k'(x) = g'(x) - f'(x)$

$= 0$

That is, the tiny line segments that link to form the graph of k are all horizontal

So $k(x) = c$ for some constant,

In particular $g(x) - f(x) = c$ or $g(x) = f(x) + C$.

This means that if you can guess one function that solves $f'(x) = \varphi(x)$, then any other solution is of the form $f(x) + C$.

For example $(\quad)' = 2x$ and if an alien

Guess $f(x) = x^2$

tells you that $g(x) =$  is also a solution, you don't have to be scared of it. $g(x) = x^2 + C$.

This will be very important later! So be sure you understand this!!!

The mathematical machinery that makes things run smoothly is Mean-Value-Theorem. Make sure you understand it!

Preview of u-substitution

In a later section, you will be doing something which by virtue of its name might feel a bit obscure.

Ex. Find $f(x)$ if $f'(x) = \cos(x^2) \cdot 2x$

Solution: **Recall:** if $b(x)$ - big fish
$s(x)$ - small fish then $\frac{d}{dx}\left(b(s(x))\right) =$
$= b'(s(x))\, s'(x)$

In our case $\underbrace{\cos}_{b'}(\underbrace{x^2}_{s(x)})\, \underbrace{2x}_{s'(x)}$ looks like

it came from chain rule. Notice that we only need to figure what was $b(x)$.

So $f(x) = \sin(x^2) + C$.

Ex. Find $f(x) = ?$ if $f'(x) = \cos(x^2) \cdot x$.

Solution: This almost looks like perfect chain rule:

$$f'(x) = \frac{1}{2} \cos(x^2)\, 2x$$

$$f(x) = \frac{1}{2} \sin(x^2) + C.$$

Before we continue, lets introduce notation.

Def: $\int f(x)\,dx$ means find all functions $F(x)$ such that $F'(x) = f(x)$. i.e. find all anti derivatives.

Ex. $\int 3x^2\,dx = x^3 + c$ $\int (5x^2 + e^x)\,dx$

$= \frac{5}{3}x^3 + e^x + c.$

Notice that $\int (f(x) + g(x))\,dx = \int f(x)\,dx + \int g(x)\,dx$

because derivative of sum is the sum of derivatives.

Similarly $\int c f(x)\,dx = c\int f(x)\,dx$

because derivatives commute with constants.

Ex. (a) $\int x^2\sqrt{x^3+1}\,dx$ (b) $\int \cos^3\theta \sin\theta\,d\theta$

(c) $\int e^x \cos(e^x)\,dx$ (d) $\int \frac{z^2}{z^3+1}\,dz$

Solution: Once you realize that you are reversing chain rule, this becomes rather easy.

(a) $\frac{1}{3}\int \boxed{3x^2}\sqrt{\underbrace{x^3+1}_{\text{inner}}}\,dx = \frac{1}{3}\frac{2}{3}\left(x^3+1\right)^{3/2} + C$

$= \frac{2}{9}(x^3+1)^{3/2} + C$

(b) $-\int (\underbrace{\cos\theta}_{\text{innner}})^3 \boxed{-\sin\theta}\,d\theta = -\frac{1}{4}\cos^4\theta + c$

(c) $\int \boxed{e^x}\cos(\underbrace{e^x}_{\text{inner}})\,dx = \sin(e^x) + c$

(d) $\frac{1}{3}\int \boxed{3z^2}\,\frac{1}{\underbrace{z^3+1}_{\text{inner}}}\,dz = \frac{1}{3}\ln|z^3+1| + c$

Ex. Place something in box to make computing antiderivative very simple.

(a) $\int \sin(x^3)\boxed{}\,dx$

(b) $\int e^{\tan^{-1}x}\boxed{}\,dx$

Solution:

(a) $\int \sin(\underbrace{x^3}_{\text{inner}})\boxed{3x^2}\,dx = -\cos(x^3) + c$

(b) $\int e^{\underbrace{\tan^{-1}x}_{\text{inner}}}\boxed{\frac{1}{1+x^2}}\,dx = e^{\tan^{-1}x} + C.$