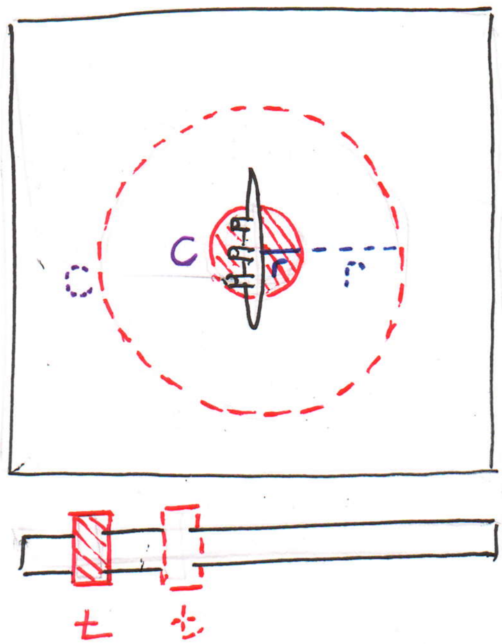
Related Rates Lecture 6.

Imagine you are watching a simplistic model of a tanker leaking oil into the sea. The water is still, the tanker isn't moving, and the oil is leaking in a circular pattern.



Q. What data changes with time?

A. Observe that the area of the spill $A(t)$, the radius of the spill $r(t)$, the diameter of the spill $d(t)$, and the circumference of the spill $C(t)$ are all parameters that change with time. They are functions of time because for each moment of time t, there is a definite area, definite radius, definite diameter and circumference. Furethe re more, note that all these parameters are related:

$$A(t) = \pi [r(t)]^2 \; ; \quad C(t) = 2\pi\, r(t) \; ; \quad C(t) = \pi d(t) \; ;$$

$$A(t) = \frac{1}{4\pi} [C(t)]^2$$

<u>Ex.</u> Suppose the tanker is leaking oil at a constant rate of 2 meters^2 per minute (it is a 2D tanker after all!)

(a) Is the rate of growth of the radius of the spill increasing, decreasing, or remains the same? That is, does the velocity with which the radius is changing change with time?

(b) If the video starts at the moment of accident, give explicit formulas for $A(t)$ and $r(t)$ as functions of time.

(c) How fast does the radius grow 5 minutes into the video?

<u>Solution:</u>

(a) The area increases at constant rate. If the velocity of expansion of the radius remained constant or grew, with each minute of time we would get at least $v \cdot t = v$ additional meters added to the radius making the area $\sim (r+v)^2 = r^2 + 2v + v^2$

In two minutes, $A \sim (r+2v)^2 = r^2 + 4v + 4v^2$

But then $A(2) - A(1) \sim (r^2+4v+4v^2) - (r^2+2v+v^2)$

$= 2v + 3v^2 > A(1) - A(0) \sim (r^2+2v+v^2) - r^2 = 2v + v^2$

(b) $A(t) = 2t$ and $r(t) = \sqrt{\dfrac{A(t)}{\pi}} = \sqrt{\dfrac{2t}{\pi}}$

(from $A = \pi r^2$).

(c) Explicitly $r'(t) = \dfrac{d}{dt}\left(\sqrt{\dfrac{2}{\pi}}\sqrt{t}\right) = \sqrt{\dfrac{2}{\pi}}\dfrac{1}{2\sqrt{t}}$

so $r'(6) = \sqrt{\dfrac{2}{\pi}}\dfrac{1}{2\sqrt{6}}$ m/min.

As was the case with implicit differentiation, it will almost always be easier to attack the problem without first solving for t.

$A(t) = \pi[r(t)]^2 \xrightarrow{\text{suppress } t} A = \pi r^2$

rates A & r are related.

$$\frac{dA}{dt} = \pi \cdot 2r \cdot \frac{dr}{dt}$$

$$2 = 2\pi r \frac{dr}{dt}$$

So $\dfrac{dr}{dt} = \dfrac{1}{\pi r}$ when $t = 6$, $A = 2 \cdot 6 = 12$

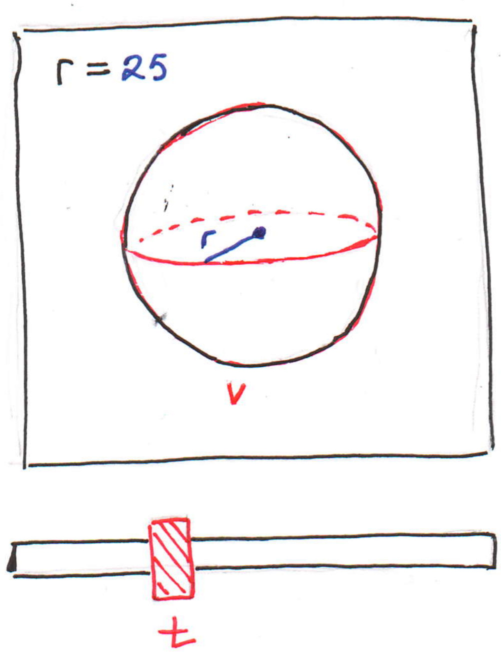
and $r = \sqrt{\dfrac{12}{\pi}}$

$$\Rightarrow \frac{dr}{dt} = \frac{1}{\sqrt{\pi}\sqrt{12}} \text{ m/min.}$$

<u>Ex.</u> Air is being pumped into a spherical balloon so that its volume increases at a rate of 100 cm^3/s. How fast is the radius of the balloon increasing when diameter is 50 cm?

<u>Solution:</u> When solving related rates problems always visualize an animation in its simplest possible form. In this case its an expanding sphere.

Make note of relevant parameters and their relationships. In this situation these are V - volume of sphere and r - its radius.



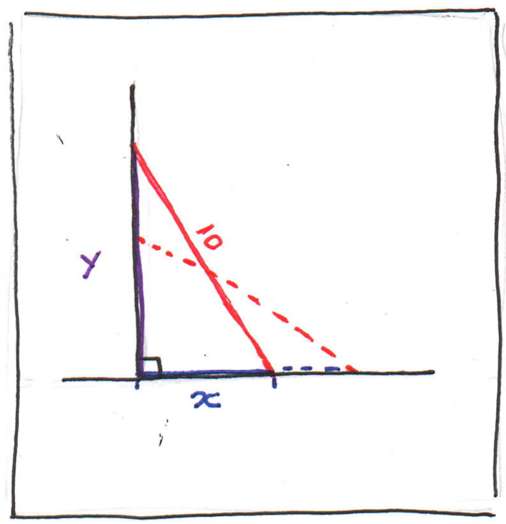
The video is paused at some moment t for which $r(t) = r = 25$. Since $V = \frac{4}{3}\pi r^3$ we have

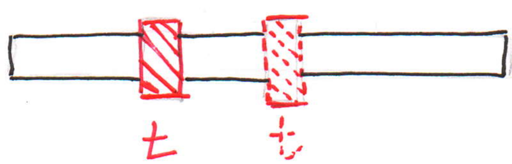
$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt} \quad \text{or} \quad 100 = 4\pi r^2 \frac{dr}{dt} \implies \frac{dr}{dt} = \frac{25}{\pi r^2}$$

at $r = 25$ this becomes $\left.\frac{dr}{dt}\right|_{r=25} = \frac{1}{\pi \cdot 25}$ cm/s.

Ex. A ladder 10 ft long rests against a vertical wall. If the bottom of the ladder slides away from the wall at a rate of 1 ft/s, how fast is the top of the ladde sliding down the wall when the bottom of the ladder is 6 ft from the wall?

Solution: What is this video really about? From the point of view of mathematics? What quantities change with time?





This video is really displaying a bunch of right triangles all of which have the same hypothenuse.

By pythagoras' theorem, $x^2+y^2=10$ and we are told that $\frac{dx}{dt} = 1$ ft/s. Thus $2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$

and $\frac{dy}{dt} = -\frac{x}{y}\frac{dx}{dt} = -\frac{x}{y}$ for all t.

The video was paused at a moment when $x = 6$

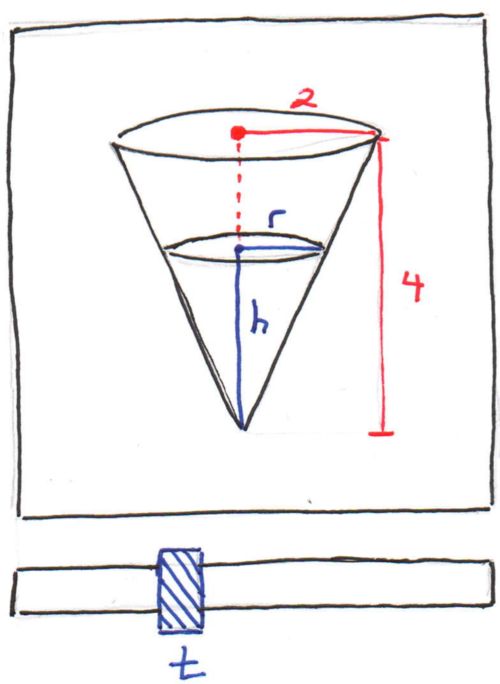
Hence $y = \sqrt{10^2 - 6^2} = \sqrt{64} = 8$.

so $\left.\frac{dy}{dt}\right|_{x=6} = -\frac{6}{8} \text{ ft/s} = -\frac{3}{4} \text{ ft/s}$.

Ex. A water tank has the shape of an inverted circular cone with base radius 2 m and height 4 m. If water is being pumped into the tank at a rate of 2 m^3/min. Find the rate at which the water level is rising when the water is 3 m deep.

Solution: The volume of a cone of height h and base radius r is $V = \frac{1}{3}\pi r^2 h$.

To mathematical eyes we are merely seeing a cone inside a larger (proportional) cone.



We are interested in relating h directly to V. Hence we need to eliminate r from $V = \frac{1}{3}\pi r^2 h$.

To do that, note that $\frac{r}{h} = \frac{2}{4}$ since the two cones are similar. Thus $r = \frac{1}{2}h$ and $V = \frac{1}{3}\pi \left(\frac{1}{2}h\right)^2 h$.

or $V = \frac{1}{12}\pi h^3$.

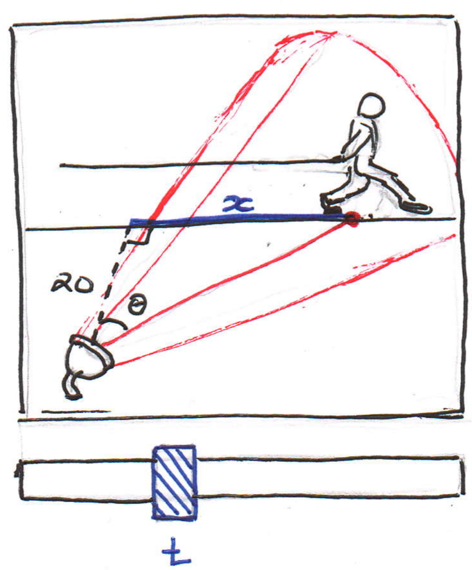
Since volume is increased at constant rate of $2\ m^3/min$, we have $2 = \frac{dV}{dt} = \frac{1}{4}\pi h^2 \frac{dh}{dt}$.

The video was paused at a moment t when $h = 3$

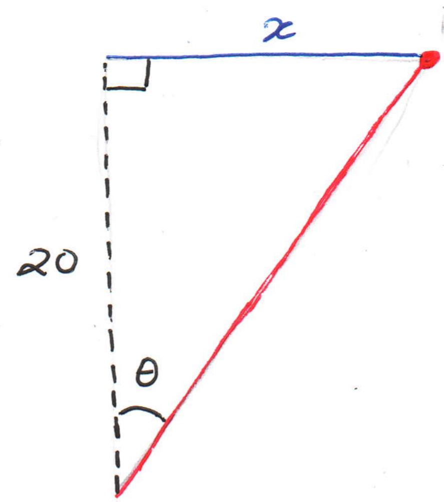
so $\frac{dh}{dt} = \frac{8\pi^{-1}}{3^2} = \frac{8}{9}\pi^{-1}\ m/min. = \frac{8}{9\pi}\ m/min \approx 0.28\ m/min.$

Ex. A man walks along a straight path at a speed of 4 ft/s. The Gesundheitsgestapo search light is located 20 ft from the path and is kept focused on the man to make sure he is a good citizen. At what rate is the searchlight rotating when the man is 15 ft from the point on the path closest to the searchlight?

Solution:



We must relate x to θ



The simplest relationship is $\tan\theta = \frac{x}{20}$

Thus $\sec^2\theta \, \frac{d\theta}{dt} = \frac{1}{20}\frac{dx}{dt} = \frac{4}{20} = \frac{1}{5}$

and $\frac{d\theta}{dt} = \frac{1}{5}\frac{1}{\sec^2\theta} = \frac{1}{5}\cos^2\theta$

We paused when $x = 15$. Hence $\cos^2\theta = \frac{20^2}{(15)^2+(20)^2}$

$= \frac{5^2 \cdot 4^2}{5^2(3^2+4^2)} = \frac{4^2}{25} = \frac{16}{25}$

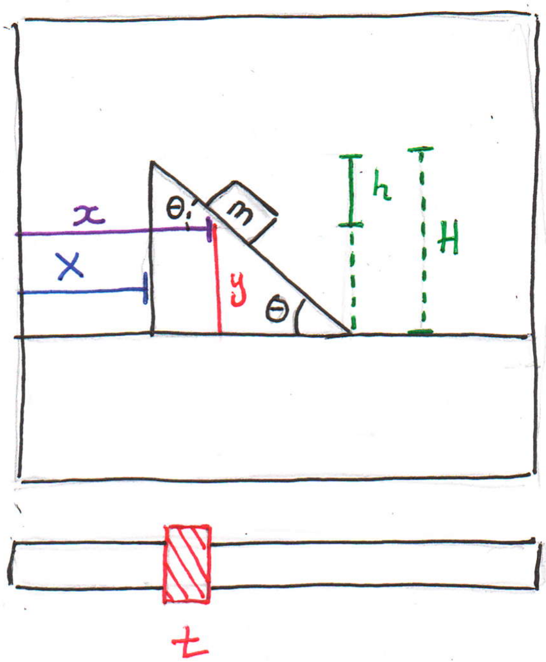
so $\left.\frac{d\theta}{dt}\right|_{x=5} = \frac{16}{125}$ rad/s $= 0.128$ rad/s.

Related rates and Mechanics

Related rates and similar ideas are very frequently used to set up kinematic constraints.

<u>Ex.</u> A block of mass m is placed on a wedge of mass M. Describe the motions in the absence of friction.

<u>Solution:</u>



The constraint for this setup is $\tan\theta = \frac{h}{x-X}$

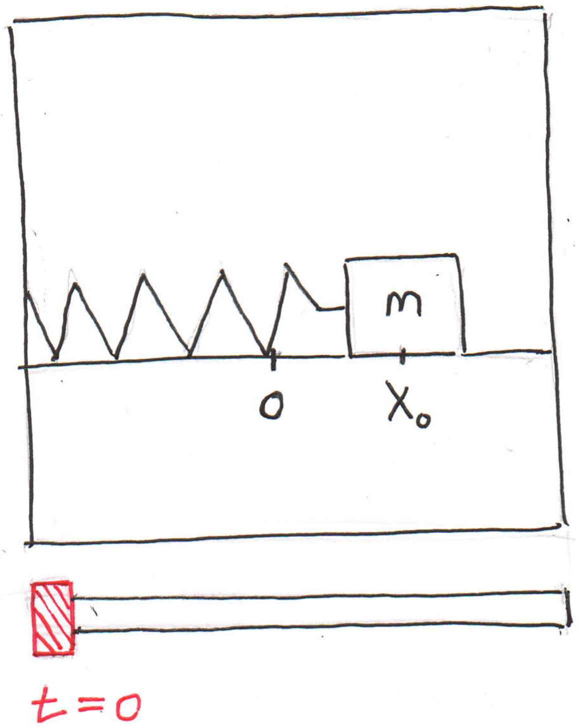
$$= \frac{H-y}{x-X}$$

Thus $(x-X)\tan\theta = H-y$ or

$(\ddot{x}-\ddot{X})\tan\theta = -\ddot{y}$ where " $\ddot{\ }$ " means second derivative with respect to time.

This is a good moment to preview the harmony of math in problems that strive to be real.

Ex. Describe the motion of the mass attached to the spring.



Solution: We wish to find a function $X(t)$ that describes the position of the block as a function of t.

By Newton's Law the force at time t satisfies

$$F(t) = M\ddot{X}(t)$$

By Hooke's Law, this force is proportional to displacement and acts in the opposite direction to displacement.

$$F(t) = -kX(t)$$

Thus $$M\ddot{X}(t) = -kX(t).$$

or $\ddot{X} + \frac{k}{M} X = 0$ (The problem is to find a function whose second derivative is $-\frac{k}{M}$ times the function).

We can write $\ddot{X} + \frac{k}{M} X = \left[\left(\frac{d}{dt}\right)^2 + \frac{k}{M} I\right] X = 0$

and this is just like a quadratic equation!

$$\left(\frac{d}{dt}\right)^2 + \frac{k}{M} I = \left(\frac{d}{dt} + i\sqrt{\frac{k}{M}}\, I\right)\left(\frac{d}{dt} - i\sqrt{\frac{k}{M}}\, I\right)$$

(This is like $T^2 + \frac{k}{m} = \left(T + i\sqrt{\frac{k}{m}}\right)\left(T - i\sqrt{\frac{k}{m}}\right)$)

Thus we ask what function $X(t)$ satisfies

$$\frac{d}{dt} X(t) = -i\sqrt{\frac{k}{M}}\, X(t) \quad \text{or} \quad \frac{d}{dt} X(t) = i\sqrt{\frac{k}{M}}\, X(t)$$

Thus $X(t) = A e^{i\sqrt{\frac{k}{m}}\, t} + B e^{-i\sqrt{\frac{k}{m}}\, t}$.

Since $\dot{X}(0) = 0$

we have $0 = i\sqrt{\frac{k}{m}}\, A e^{i\sqrt{\frac{k}{m}} \cdot 0} + \left(-i\sqrt{\frac{k}{m}}\, B e^{-i\sqrt{\frac{k}{m}} \cdot 0}\right)$

or $0 = A - B$

while $X(0) = X_0 = A + B = 2A$.

so $A = \frac{X_0}{2}$.

Observe that $e^{i\sqrt{\frac{k}{m}}\, t} = \cos\left(\sqrt{\frac{k}{m}}\, t\right) + i \sin\left(\sqrt{\frac{k}{m}}\, t\right)$

and $e^{-i\sqrt{\frac{k}{m}}\, t} = \cos\left(\sqrt{\frac{k}{m}}\, t\right) - i \sin\left(\sqrt{\frac{k}{m}}\, t\right)$

Thus $x(t) = A\left(e^{-i\sqrt{\frac{k}{m}}t} + e^{i\sqrt{\frac{k}{m}}t}\right) = 2A\cos\left(\sqrt{\frac{k}{m}}\,t\right)$

$= x_0 \cos\left(\sqrt{\frac{k}{m}}\,t\right).$

Just look at the number of ideas we went through in order to solve this problem!