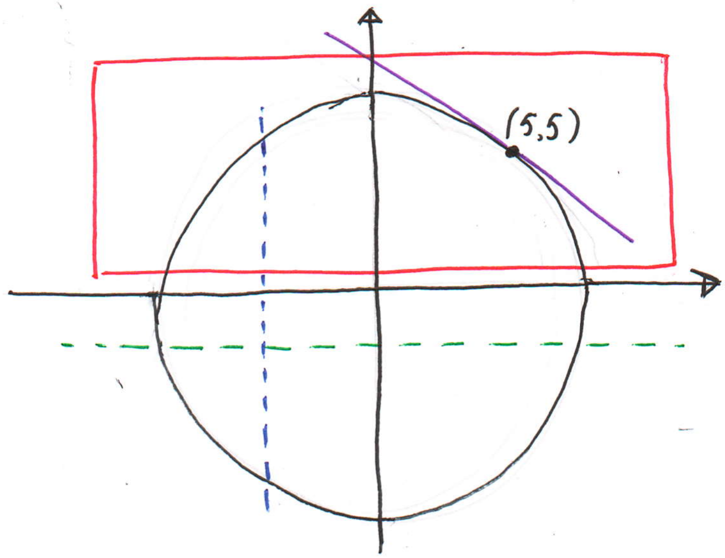
Implicit Differentiation Lecture 3

By now you are hopefully used to the idea of differentiating functions with definite formulas. But suppose you are asked to find the slope of the tangent line at a point on a circle.



Circle $x^2+y^2=25$

The equation of the circle does not determine y as a global function of x, because the circle does not pass the vertical line test. Neither can x be a global function of y, because the circle doesn't pass the horizontal line test.

Nevertheless, it seems that as we zoom in on a point of the circle, the curve looks more and more like a line. (Our planet is a constant evidence for this!).

Suppose we wish to describe a tangent to the circle in the upper-half plane.

In particular, let's zoom in on the point (5,5), what can we say about the tangent line there?

Method 1: Explicit Differentiation

Notice that the tangent line is at a point in the upper-half plane where $y>0$. We can solve explicitly for y using this constraint,

$$x^2+y^2=25 \implies y^2 = 25-x^2 \implies y = \pm\sqrt{25-x^2}$$

$$\implies y = \sqrt{25-x^2}$$

Please note that $y = \boxed{y(x) = \sqrt{25-x^2}}$ is an explicit function of x.

$$y' = \frac{1}{2\sqrt{25-x^2}} \cdot (-2x) = \frac{-x}{\boxed{\sqrt{25-x^2}}} = \frac{-x}{y}.$$

Now $y'\Big|_{(5,5)} = \frac{-5}{5} = -1.$

Method 2: Pretend that you have explicitly solved for y. You have $y=y(x)$, which means that you're imagining you're staring at a formula of x which you know how to differentiate.

$$\frac{d}{dx}\left(x^2+[y(x)]^2\right) = \frac{d}{dx}(25)$$

$$\underbrace{2x + 2y(x)}\cdot \underbrace{y'(x)} = 0$$

$$\left(\square^2\right)' = 2\square \cdot \square'$$

(Notice that we don't know how to differentiate y. We just imagine that it is differentiable and work from there.)

$$y'(x) = \frac{-2x}{2y(x)} = \frac{-2x}{2y} = \frac{-x}{y}$$

$$y'(5) = \frac{-5}{5} = -1.$$

We have arrived at the same solution by entirely side stepping the process of finding a definite formula for $y = y(x)$.

<u>Ex.</u> Find an equation of the tangent line through the point $(1,2)$ for the curve determined by the equation

$$y^2 + 4x^2 = 4xy.$$

<u>Solution:</u> <u>Method 1:</u> Explicitly.

$y^2 + 4x^2 - 4xy = y^2 + (-4xy) + 4x^2 = 0$ This is a perfect square $(y-2x)^2 = 0$. Thus $y = 2x$.

and $y' = 2$. Naturally a line is its own tangent everywhere

The tangent line is therefore $y = 2x$.

<u>Method 2:</u> Implicitly. We imagine that we already know the formula $y = y(x)$ and work from there.

$$\frac{d}{dx}\left(y^2 + 4x^2\right) = \frac{d}{dx}(4xy)$$

$$\underbrace{2y \cdot y'} + 8x = \underbrace{4y + 4x \cdot y'}$$

Chain Rule

$$\left([y(x)]^2\right)' = 2y(x) \cdot y'(x)$$

Suppress $y(x)$ as y. Fewer symbols make for faster calculations!

Product Rule combined with chain Rule

$$[4x\,y(x)]' = 4y(x) + 4x\,y'(x).$$

Again, we don't know formula $y(x)$ so we write $y'(x)$ as symbolic diff.

$$\underbrace{2yy' - 4xy' = 4y - 8x}$$

Bring all y' to one side of equation.

$$(2y - 4x)y' = 4y - 8x$$

$$y' = \frac{4y - 8x}{2y - 4x} = 2.$$

As it happens, most equations that appear in scientific investigations are difficult or outright impossible to solve explicitly.

Ex. Find an equation of the tangent line to the curve $x^3+y^3=6xy$, at the point $(3,3)$

Solution: We can solve this equation for $y=y(x)$ using the cubic formula. If we go through the trouble we will obtain

$$y(x)=\sqrt[3]{-\frac{1}{2}x^3+\sqrt{\frac{1}{4}x^6-8x^3}}+\sqrt[3]{-\frac{1}{2}x^3-\sqrt{\frac{1}{4}x^6-8x^3}}$$

which is still very difficult to differentiate even after all the labor to obtain this formula...

It is much easier to imagine that we know the formula and work from there.

$$\frac{d}{dx}\left(x^3+y^3\right)=\frac{d}{dx}(6xy)$$

$$3x^2+3y^2\cdot y'=6y+6xy'$$

$$\left(3y^2-6x\right)y'=6y-3x^2$$

$$y'=\left.\frac{6y-3x^2}{3y^2-6x}\right|_{(3,3)}=\frac{6\cdot 3-3\cdot 3^2}{3\cdot 3^2-6\cdot 3}=-1.$$

Thus an equation of the tangent line is given by

$$y-3=-1(x-3).$$

Ex. Find y' if $\sin(x+y)=y^2\cos x$

Solution: Solving explicitly for y would be painfully difficult! impossible even if we have too few functions in our tool kit. However, we can side step the process of solving for y.

$$\frac{d}{dx}\left(\sin(x+y)\right)=\frac{d}{dx}\left(y^2\cos x\right)$$

$$\cos(x+y)\cdot(1+y')=2y\cdot y'\cos x+y^2(-\sin x)$$

$$\cos(x+y)+\cos(x+y)y'=(2y\cos x)y'-y^2\sin x$$

$$\left(\cos(x+y)-2y\cos x\right)y'=-y^2\sin x-\cos(x+y)$$

Thus $$y'=\frac{-y^2\sin x-\cos(x+y)}{\cos(x+y)-2y\cos x}.$$

Now it's your turn! Solve the following problems.

<u>Ex.</u> Find $\frac{dy}{dx}$.

(a) $2x^3 + x^2y - xy^3 = 2$

(b) $xe^y = x - y$

(c) $\sqrt{x+y} = 1 + x^2y^2$

(d) $x \sin y + y \sin x = 1$

(e) $\tan(x-y) = \frac{y}{1+x^2}$

(f) $y \sin 2x = x \cos 2y$.

<u>Solution:</u> Let's compete! See who can solve it faster!

(a) $$y' = \frac{-(6x^2+2xy-y^3)}{x^2-3xy^2} \quad \text{Done!}$$

Here it is in slow motion.

$$(2x^3 + x^2y - xy^3)' = 0 \Rightarrow 6x^2 + 2xy + \underline{x^2y'} - y^3 -$$

$$\underline{-3xy^2y'} = 0 \Rightarrow (x^2 - 3xy^2)y' = -6x^2 - 2xy + y^2$$

$$\Rightarrow \boxed{y' = \frac{-6x^2-2xy+y^2}{x^2-3xy^2}.}$$

(b) $$y' = \frac{-(e^y-1)}{xe^y+1} \quad \text{Done!}$$

Here it is in slow motion.

$$(xe^y)' = (x-y)' \Rightarrow e^y + \underline{xe^y y'} = 1 - \underline{y'}$$

$$(xe^y + 1)y' = 1 - e^y$$

$$\boxed{y' = \frac{1-e^y}{xe^y+1}}$$

(c) $$y' = \frac{-\left(\frac{1}{2\sqrt{x+y}} - 2xy^2\right)}{\left(\frac{1}{2\sqrt{x+y}} - 2x^2y\right)} = -\frac{1-4xy^2\sqrt{x+y}}{1-4x^2y\sqrt{x+y}}$$

Here are the details

$$\left(\sqrt{x+y}\right)' = (1+x^2y^2)'$$

$$\frac{1}{2\sqrt{x+y}}(1+y') = 2xy^2 + 2x^2y \cdot y'$$

$$\left(\frac{1}{2\sqrt{x+y}} - 2x^2y\right)y' = 2xy^2 - \frac{1}{2\sqrt{x+y}}$$

$$\boxed{y' = \frac{2xy^2 - \frac{1}{2\sqrt{x+y}}}{\frac{1}{2\sqrt{x+y}} - 2x^2y}}$$

(d) $$y' = \frac{-(\sin y + y\cos x)}{x\cos y + \sin x}$$ Done!

$$(x\sin y + y\sin x)' = 0$$

$$\sin y + \underline{x\cos y \cdot y'} + \underline{y'\sin x} + y\cos x = 0$$

$$(x\cos y + \sin x)\,y' = -y\cos x - \sin y$$

$$\boxed{y' = \frac{-\sin y - y\cos x}{x\cos y + \sin x}}$$

(e) $$y' = \frac{-\left(\sec^2(x-y) + \frac{2xy}{(1+x^2)^2}\right)}{\left(-\sec^2(x-y) - \frac{1}{1+x^2}\right)}$$ Done!

$$(\tan(x-y))' = \left(\frac{y}{1+x^2}\right)'$$

$$\sec^2(x-y)\,(1-y') = \frac{(1+x^2)y' - 2xy}{(1+x^2)^2}$$

$$\sec^2(x-y) - \underline{\sec^2(x-y)\,y'} = \underline{\frac{1}{1+x^2}\,y'} - \frac{2xy}{(1+x^2)^2}$$

$$\left(-\sec^2(x-y) - \frac{1}{1+x^2}\right) y' = -\sec^2(x-y) - \frac{2xy}{(1+x^2)^2}$$

$$y' = \frac{-\sec^2(x-y) - \frac{2xy}{(1+x^2)^2}}{-\sec^2(x-y) - \frac{1}{1+x^2}}$$

$$= \frac{-(1+x^2)^2\sec^2(x-y) - 2xy}{-(1+x^2)^2\sec^2(x-y) - (1+x^2)}.$$

(f) $$y' = \frac{-(2y\cos 2x - \cos 2y)}{\sin 2x + 2x\sin 2y}$$

$$(y\sin 2x)' = (x\cos 2y)'$$

$$y'\sin 2x + 2y\cos 2x = \cos 2y - 2x\sin 2y\, y'$$

$$(\sin 2x + 2x\sin 2y)\,y' = \cos 2y - 2y\cos 2x$$

or $$\boxed{y' = \frac{\cos 2y - 2y\cos 2x}{\sin 2x + 2x\sin 2y}}.$$

How am I able to solve these problems so fast? Well, the truth of the matter is that I applied the much more efficient techniques from Calculus III. We will not have the time to develop these ideas adequetly enough for you to understand why the trick works, but I will reveal it in hope that it provides you with a useful tool to quickly check your work.

Implicit differentiation a la Calc. III

First observe that every equation we differentiated implicitly is of the form $F(x,y) = 0$, where $F(x,y)$ is a function of two variables.

Ex. Identify $f(x,y)$ in the following equations:

(a) $x^2+y^2=36$

(b) $x^2+xy-y^2=4$

(c) $e^x \cos x = 1+\sin(xy)$

(d) $e^{x/y} = x-y$

(e) $\cos(xy) = 1+\sin y$

(f) $\tan(x-y) = \dfrac{y}{1+x^2}$

Solution: We move everything to one side of the equation.

(a) $\underbrace{x^2+y^2-36}_{f(x,y)} = 0$

Thus $f(x,y) = x^2+y^2-36$. For instance $f(6,6) = 6^2+6^2-36 = 36$.

(b) $\underbrace{x^2+xy-y^2-4}_{f(x,y)} = 0$

Thus $f(x,y) = x^2+xy-y^2-4$

(c) Clearly $f(x,y) = e^x \cos x - \sin(xy) - 1$ works.

(d) $f(x,y) = e^{x/y} - x + y$

(e) $f(x,y) = \cos(xy) - \sin y - 1$

(f) $f(x,y) = \tan(x-y) - \dfrac{y}{1+x^2}$.

By Calc. III Chain Rule

$\frac{d}{dx} f(x, y(x)) = \frac{d}{dx}(0) = 0$

$\frac{d}{dx} f(x, y(x)) = f_x(x,y) + f_y(x,y)\frac{dy}{dx} = 0$ where

f_x and f_y are the partial derivatives with respect to x and y.

Thus $\boxed{\frac{dy}{dx} = \frac{-f_x(x,y)}{f_y(x,y)}}$

Don't understand what this means? Don't worry!

f_x means: treat y as constant (imagine y is like 5) and differentiate with respect to x.

f_y means: treat x as constant (imagine x is like 5) and differentiate with respect to y.

Ex. See if you can find $\frac{dy}{dx}$ using the Calc. III method.

(a) $e^y \cos x = 1 + \sin(xy)$

(b) $x^2 + y^2 = 1$

(c) $xe^y = x - y$

(d) $4\cos x \sin y = 1$

Solution:

(a) $f(x,y) = e^y \cos x - \sin(xy) - 1 = 0$

$f_x = -e^y \sin x - \cos(xy)\cdot y$

$f_y = e^y \cos x - \cos(xy)\cdot x$

$$\frac{dy}{dx} = -\frac{-e^y \sin x - \cos(xy)\cdot y}{e^y \cos x - \cos(xy)\cdot x}$$

(b) $f(x,y) = x^2 + y^2 - 1$

$f_x = 2x$

$f_y = 2y$

$$\frac{dy}{dx} = -\frac{2x}{2y} = -\frac{x}{y}.$$

(c) $f(x,y) = xe^y - x + y = 0$

$f_x = e^y - 1$

$f_y = xe^y + 1$

$$\frac{dy}{dx} = -\frac{e^y - 1}{xe^y + 1}$$

(d) $f(x,y) = 4\cos x \sin y - 1 = 0$

$f_x = -4 \sin x \sin y$

$f_y = 4 \cos x \cos y$

$$\frac{dy}{dx} = -\frac{-4\sin x \sin y}{4\cos x \cos y} = \tan x \tan y.$$

Be careful! You can use the Calc. III method to check your work, but it is likely that you don't understand at the moment ~~that~~ why the method works.
What you don't understand can easily confuse you and cause you to make mistakes.
Now let's be a bit more critical about implicit differentiation.

Ex. Find $\frac{dy}{dx}$ given $x^2-2xy+y^2=-10$

Solution: $2x-2y-\underline{2xy'}+\underline{2y\cdot y'}=0$

$(2y-2x)y' = 2y-2x$

$y' = 1.$

Right? Wrrrong!!!

$x^2-2xy+y^2=(x-y)^2=-10$. But this equation has no real solutions because a number squared is nonnegative.
If we don't solve the equation, how do we know it can be solved?

You do not need to worry about it in this class but there is a deep theorem in multivariable calculus that assures us that our work isn't in vain even when we cannot solve the equation.

<u>Thm:</u> (Implicit function theorem). Let $f: \mathbb{R}^2 \to \mathbb{R}$ be a continuously differentiable function.
If the equation $f(x,y) = 0$ has a solution (a,b) and $\dfrac{-f_x(a,b)}{f_y(a,b)}$ is defined, then $y = y(x)$ is a function of x for x near a and $y'(a) = -\dfrac{f_x(a,b)}{f_y(a,b)}$.

<u>Ex.</u> The equation $\sin(\pi xy) + xy = 3y$ has a solution $x = 1$ $y = \frac{1}{2}$

Since $f_y\Big|_{(1,\frac{1}{2})} = \pi x \cos(\pi xy) + x - 3\Big|_{(1,\frac{1}{2})} = -2 \neq 0$. Thus $y = y(x)$ is locally a function of x near $x = 1$.

<u>Ex.</u> Find an equation of the tangent line to the curve $y \sin 2x = x \cos 2y$ at $\left(\frac{\pi}{2}, \frac{\pi}{4}\right)$

<u>Solution:</u> $y' \sin 2x + 2y \cos 2x = \cos 2y + 2x(-\sin 2y) y'$

There is no need to solve the general equation.

Plug in $(\frac{\pi}{2}, \frac{\pi}{4})$,

$$y' \cdot 0 + 2 \cdot \frac{\pi}{4}(-1) = 0 + \pi(-1) y'$$

so $$-\frac{\pi}{2} = -\pi y'$$

$$y' = \frac{1}{2}.$$