Chain Rule Lecture 2

Before moving on, let's check that your knowledge of logarithms is up to date.

Review of logarithms

Given a positive number a, multiplication of a by itself a number of times has a convenient notation.

$a \cdot a \cdot a = a^3$ (e.g. $5 \cdot 5 \cdot 5 = 5^3$)

In general $\underset{1}{a} \cdot \underset{2}{a} \cdot \underset{3}{a} \cdot \ldots \cdot \underset{n}{a} = a^n$. a^n is just a convenient way to keep account of the number of multiplications.

1) $a^n \cdot a^m = a^{n+m}$ for every pair of positive integers $n, m \in \mathbb{N}$. Why?

$$a^3 \cdot a^4 = \overbrace{(\underset{1}{a} \cdot \underset{2}{a} \cdot \underset{3}{a})(\underset{1}{a} \cdot \underset{2}{a} \cdot \underset{3}{a} \cdot \underset{4}{a})}^{\text{together } 3+4} = a^{3+4} = a^7.$$

It is easy to see that this is true in general.

2) $\dfrac{a^n}{a^m} = a^{n-m}$. To see why, assume for a moment that n, m are positive integers with $n \geq m$

e.g. $$\frac{a^4}{a^3} = \frac{\cancel{a} \cdot \cancel{a} \cdot \cancel{a} \cdot a}{\cancel{a} \cdot \cancel{a} \cdot \cancel{a}} = a^{4-3}$$ (only a^{4-3} a's survive in the numerator).

If we want the property $\frac{a^n}{a^m} = a^{n-m}$ to continue to hold even when $n<m$, we have to define $a^{-k} = \frac{1}{a^k}$ for any positive $k>0$. Why?

$$\frac{a^3}{a^4} = \frac{\cancel{a}\cdot\cancel{a}\cdot\cancel{a}}{\cancel{a}\cdot\cancel{a}\cdot\cancel{a}\cdot a} = \frac{1}{a}.$$

Only as in denominator survive!

3) $a^0 = 1$ because $a^0 = a^{1-1} = \frac{a}{a} = 1.$

Comprehension Check

What is 0^0? Why?

4) $(a^n)^m = a^{nm}$.

This can be seen from a concrete example

$$(a^2)^3 = a^2\cdot a^2\cdot a^2 = (a\cdot a)(a\cdot a)(a\cdot a) = a^6 = a^{2\cdot 3}$$

In general $(a^n)^m =$

$$m\left[\overbrace{\begin{matrix} a\cdot a\cdot\ldots\cdot a \\ a\cdot a\cdot\ldots\cdot a \\ \vdots \qquad\quad \vdots \\ a\cdot a\cdot\ldots\cdot a \end{matrix}}^{n}\right. = a^{m\cdot n}$$

Do you see?

5) $\sqrt[n]{a}$ is represented by $a^{\frac{1}{n}}$;

$a^x = \sqrt[n]{a} \Rightarrow (a^x)^n = (\sqrt[n]{a})^n = a$. If we wish $(a^x)^y = a^{xy}$ to be a property that continues to hold for non integer values x and y

then $(a^x)^n = a^{x \cdot n} = a^1 \Rightarrow x \cdot n = 1 \Rightarrow x = \frac{1}{n}$.

I will spare you the torture, but we go on to define

$a^{\frac{m}{n}} = (a^{\frac{1}{n}})^m$. To make this definition logically sound, many heighly nontrivial verifications must be carried. For instance, it is not obvious that

$$a^{\frac{1}{n}} = a^{\frac{k}{kn}} \quad (\text{e.g. } 5^{\frac{1}{3}} \stackrel{?}{=} 5^{\frac{2}{6}})$$

Such considerations lead to interesting realizations.

Ex. what is $1^{1/3}$?

Solution: In this course you can answer bravely

$1 \cdot 1 \cdot 1 = 1^3 = 1$ so $1^{1/3} = \sqrt[3]{1} \equiv$ a number that when multiplied by itself $= 1 \Rightarrow$ this number $= 1$

Simply $\sqrt[3]{1} = 1$. But...

$$1^{1/3} = \begin{cases} 1 \\ -\frac{1}{2} + \frac{\sqrt{3}}{2} i \\ -\frac{1}{2} - \frac{\sqrt{3}}{2} i \end{cases}$$

Now let us talk about logarithms. Can you solve these equations?

(a) $2^x = 4$

(b) $2^x = 8$

(c) $2^x = 1$

(d) $2^x = \frac{1}{2}$

(e) $2^x = \frac{1}{4}$

(f) $2^x = \frac{1}{8}$

(g) $2^x = \sqrt{2}$

(h) $2^x = \sqrt[5]{2}$

(i) $2^x = \sqrt[7]{\frac{1}{2}}$

Solutions:

(a) $2 \cdot 2 = 4$ $\quad 2^2 = 4$ so $x = 2$ (Guess)

(b) $2 \cdot 2 \cdot 2 = 8$ $\quad 2^3 = 8$ so $x = 3$ (Guess)

(c) $2^0 = 1$ so $x = 0$ (Guess)

(d) $2^{-1} = \frac{1}{2}$ so $x = -1$ (Guess)

(e) $2^{-2} = \frac{1}{4}$ so $x = -2$ (Guess)

(f) $2^{-3} = \frac{1}{8}$ so $x = -3$ (Guess)

(g) $2^{\frac{1}{2}} = \sqrt{2}$ so $x = \frac{1}{2}$ (Guess)

(h) $2^{\frac{1}{5}} = \sqrt[5]{2}$ so $x = \frac{1}{5}$ (Guess)

(i) $2^{-\frac{1}{7}} = \sqrt[7]{\frac{1}{2}}$ so $x = -\frac{1}{7}$ (Guess).

How about this equation?

$2^x = -4$

Solution:

$2^0 = 1$ doesn't work! If $x > 0$ 2^x is some multiplication of 2 (or its roots) and is therefore a positive number. If $x < 0$ $2^x = \left(\frac{1}{2}\right)^{-x}$ where $-x > 0$.
Therefore 2^x is some multiplication of $\left(\frac{1}{2}\right)$ (or its roots) and is a positive number.
There is no real number x such that 2^x is a negative number.
Up until this moment we simply guessed the solution. What would you do if asked to solve the equation $2^x = 5$? It is no longer simple to guess the value of x! Remarkably, a very important step in solving many problems explicitly is to name the solution. That is, pretend that you solved it.

$2^x = 5 \Rightarrow x = \log_2 5.$

Def: $a^x = b \Rightarrow x = \log_a b$. That is, the number that placed as the power of a and thereby yields b.

Think of Alice in Wonderland. and the vial she used to shrink.

you don't have to think about the mechanism. Just follow instruction and the effect is achieved.

Ex. Solve

(a) $10^x = 3$

(b) $6^x = 2$

(c) $5^x = 7$

(d) $9^x = 3$

Solution:

(a) $10^x = 3 \Rightarrow x = \log_{10} 3$ (This is like the tag on the vial that reads: place me over 10 and get 3)

(b) $6^x = 2 \Rightarrow x = \log_6 2$ (place me over 6 and get 2)

(c) $5^x = 7 \Rightarrow x = \log_5 7$

(d) $9^x = 3 \Rightarrow x = \log_9 3$ (in this case we can identify $\log_9 3 = \frac{1}{2}$ because we know an explicit number that works).

Difference between functions with explicit formulas and the rest

Many students find the idea of solving equations implicitly difficult to wrap their heads around (can one wrap one's head around? Is it a thing?) But this is what they have been doing all the time! For example we solve equations of the form $x^2-2=0$ by saying $x=+\sqrt{2},-\sqrt{2}$. But what is $\sqrt{2}$? $\sqrt{2}$ is just a name!! It is like a tag on the vail that reads: multiply me by myself and get 2. This description doesn't explicitly talk about the number.

Comprehension Check

What is a notable difference between polynomials and functions like $\tan x$, $\sin x$, e^x, $\log_a x$, $\sqrt{x}$ etc.

Solution: Take some polynomial, $p(x)=1-5x+x^2$ for instance. Then $p(x)$ describes explicitly how the input x is transformed (digested) by p.

For instance $p(-1) = 1-5(-1)+(-1)^2 = 1+5+1 = 7$.
students subconsciously seek such explicit patterns
For instance many make this common mistake:

$$\boxed{\frac{\sin(x^2)}{x} = \sin(x)}$$

But stop and think!!! Why and when cancellation of terms works?

$\sin(x) \overset{!!!}{\neq} \sin \cdot x$, Sin is a code name for a complex command. The name gives no clue at all about the arithmetic operations with x that would describe what Sin is doing to x. (Do they ever exist?)

You might think that functions like 2^x are explicit. After all $2^3 = 2\cdot 2\cdot 2 = 8$ - a process that describes arithmetically what is explicitly happening to 3.
However $2^{1/2} = \sqrt{2}$. How would you describe this by an explicit arithmetic operation?
Recall that we were able to establish that

$$e^x = 1 + x + \tfrac{1}{2}x^2 + \tfrac{1}{6}x^3 + \ldots + \tfrac{1}{n!}x^n + \ldots +$$

This can be paraphrased by saying that e^x is an infinite polynomial; e^x is an explicit algebraic operation that processes (I like the word verarbeitet) x into $1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \ldots +$.

In practice, we truncate the infinite polynomial and get an approximation.

Can you find such polynomial process for $\sin x$ and $\cos x$?

It is a similar story with $\log_a x$. $\log_a x$ doesn't describe what is happening with x arithmetically.

Properties of Logarithms

The expression $\log_a x$ means: place me over a and get x.

Comprehension Check:

What is $5^{\log_5 \text{☺}}$?

<u>Solution:</u> Logically

$5^{\log_5 ☺} = ☺$ because $\log_5 ☺$ means

"place me over 5 and get ☺".

One has to be careful!

<u>Ex.</u> $5^{\log_5(-7)} = ?$

<u>Solution:</u> It is tempting to say that $5^{\log_5(-7)} = -7$

However, no real power of 5 will yield a negative number.

It is something we can explain later (by means of intermediate value theorem), but $\log_a x$ makes sense for every $a>0$ and every $x>0$.

$\log_a x$ means "I am the unique power of a that placed over a gives x".

We observe the following important logarithmic properties:

1) $\log_a(M \cdot N) = \log_a M + \log_a N$

2) $\log_a\left(\frac{M}{N}\right) = \log_a M - \log_a N$

3) $\log_a(M^r) = r\log_a M$.

4) (Change of basis) $\log_b M = \dfrac{\log_a M}{\log_a b}$

proof:

1) $a^{\log_a(MN)} = \boxed{M}\boxed{N} = \boxed{a^{\log_a M}}\boxed{a^{\log_a N}}$

$= a^{\log_a M + \log_a N}$

Thus $\log_a MN = \log_a M + \log_a N$

2) Now's your turn! Carry out a similar calculation.

3) $a^{\log_a(M^r)} = M^r = \left(a^{\log_a M}\right)^r = a^{r \log_a M}$

Thus $\log_a M^r = r \log_a M$

4) $\boxed{b^{\log_b M}} = \left(a^{\log_a b}\right)^{\log_b M} = a^{(\log_a b)(\log_b M)}$

$= \boxed{M} = a^{\log_a M}$

Thus $(\log_a b)(\log_b M) = \log_a M$ or

$$\log_b M = \frac{\log_a M}{\log_a b}.$$

Derivatives of exponential functions

logarithm base e is so important that we call it the natural logarithm and write $\ln x$ instead of $\log_e x$.

Recall how we differentiated 2^x:

$$\frac{d}{dx}(2^x) = \lim_{h\to 0} \frac{2^{x+h}-2^x}{h} = \lim_{h\to 0} \frac{2^x 2^h - 2^x}{h}$$

$$= \lim_{h\to 0} 2^x \frac{2^h-1}{h} \approx 2^x \cdot \frac{2^{0.01}-1}{0.01} \approx 2^x(0.696)$$

We can do something better!

$$\frac{d}{dx}(2^x) = \frac{d}{dx}\left(e^{\ln 2^x}\right) = \frac{d}{dx}\left(e^{(\ln 2)x}\right)$$

$$= e^{(\ln 2)x} \cdot ([\ln 2]x)' = e^{(\ln 2)x} \cdot \ln 2$$

$$= e^{\ln 2^x} \cdot (\ln 2) = 2^x \cdot \ln 2.$$

And in general,

$$\frac{d}{dx}(a^x) = \frac{d}{dx} e^{x\ln a} = e^{x \ln a} \ln a = a^x \ln a$$

Observe that $\boxed{\lim_{h\to 0} \frac{a^h-1}{h} = \ln a}$

Ex. Find the derivatives

(a) $\frac{d}{dx}\left(5^{\sec x}\right)$

(b) $\frac{d}{dx}\left(2^{x^2}\cdot x^3\right)$

(c) $\frac{d}{dx}\left(\sec(5^x)\right)$

(d) $\frac{d}{dx}\left(\tan(3^x)+3^{\tan x}\right)$

Solution:

(a) $\frac{d}{dx}\left(5^{\sec x}\right) = 5^{\sec x}\cdot \ln 5 \cdot \sec x \tan x$

(b) $\frac{d}{dx}\left(2^{x^2}\cdot x^3\right) = 2^{x^2}\cdot \ln 2 \cdot 2x \cdot x^3 + 2^{x^2}\cdot 3x^2$

(c) $\frac{d}{dx}\left(\sec(5^x)\right) = \sec(5^x)\tan(5^x)\cdot 5^x \ln 5$

(d) $\frac{d}{dx}\left(\tan(3^x)+3^{\tan x}\right) = \sec^2(3^x)\cdot 3^x \ln 3 +$

$+\, 3^{\tan x}\cdot \ln 3 \cdot \sec^2 x.$