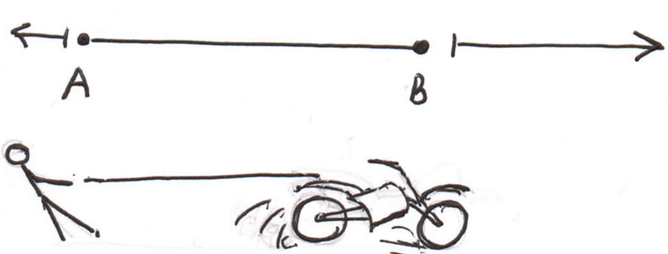
Trigonometry Lecture 4

When things get a bit difficult, it is tempting to tune out. "Why do I need this?" "Why bother?" "Why make something easy <u>unnecessarily difficult</u>?" – Students that think the latter might one day bring my end due to apoplectic shock.

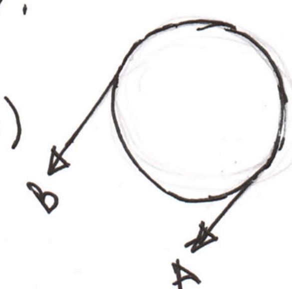
Here is an example in which calculus methods come naturally about. Don't worry if you don't understand it. I'm just trying to show you the extent of knowledge and comfort with the techniques that you will need to reach if you wish to solve even <u>simple</u> applied problems.

<u>Ex.</u> (Capstan). You are trying to stop a motorcycle by pulling on a chain attached to the motorcycle. Which strategy is more effective? Why?

(a)

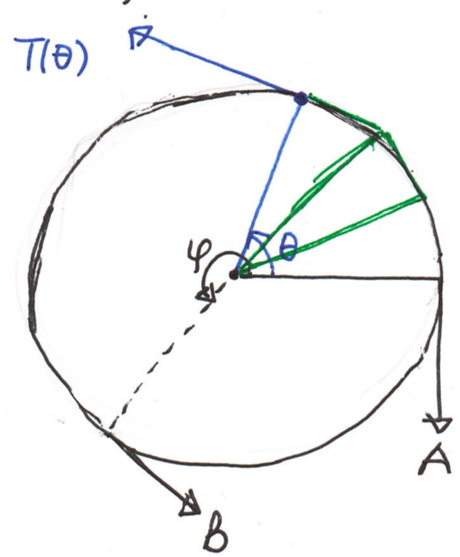


(b)



chain wrapped around pole.

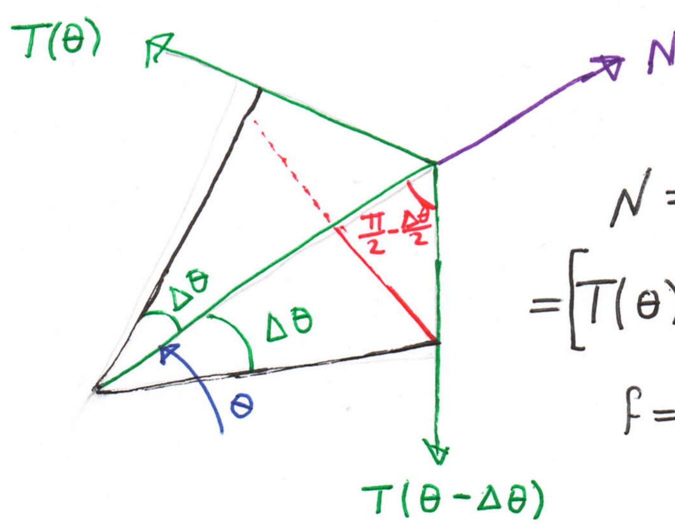
Let A be the force you exert on motorcycle and B be the force the motorcycle exerts on you. Unless you are the hulk $A \ll B$ (much smaller)



φ - angle where chain contacts the poll.

μ - coefficient of friction.

For $0 \leq \theta \leq \varphi$ let $T(\theta)$ be the tension in the chain at θ and f be the friction. If the motorcycle and you are static, $T(\theta)$ satisfies the following local property



$$N = \left[T(\theta) + T(\theta - \Delta\theta)\right] \cos\left(\frac{\pi}{2} - \frac{\Delta\theta}{2}\right)$$

$$= \left[T(\theta) + T(\theta - \Delta\theta)\right] \sin\left(\frac{\Delta\theta}{2}\right)$$

$$f = \mu N$$

and $\left[T(\theta) - T(\theta - \Delta\theta)\right] \sin\left(\frac{\pi}{2} - \frac{\Delta\theta}{2}\right) - f = 0$

Thus we get

$$\left[T(\theta) - T(\theta-\Delta\theta)\right] \cos\left(\frac{\Delta\theta}{2}\right) = f = \mu N =$$

$$= \mu\left[T(\theta) + T(\theta - \Delta\theta)\right] \sin\left(\frac{\Delta\theta}{2}\right)$$

or $$\frac{T(\theta) - T(\theta-\Delta\theta)}{\tan\left(\frac{\Delta\theta}{2}\right)} = \mu\left[T(\theta) + T(\theta-\Delta\theta)\right]$$

Letting $h = \Delta\theta$ we see that

$$\lim_{h \to 0} \frac{T(\theta) - T(\theta-h)}{\tan\left(\frac{h}{2}\right)} = \mu \lim_{h \to 0} \left[T(\theta) + T(\theta - h)\right]$$

What are these limits in terms of T?

Well...

$\lim_{h \to 0} T(\theta) + T(\theta - h) = 2T(\theta)$ because T is continuous.

How about $\lim_{h \to 0} \frac{T(\theta) - T(\theta-h)}{\tan\left(\frac{h}{2}\right)}$?

$$\lim_{h \to 0} \frac{T(\theta) - T(\theta - h)}{\tan(\frac{h}{2})} = \lim_{h \to 0} \frac{T(\theta) - T(\theta - h)}{\boxed{\frac{1}{2}h}} \cdot \frac{\boxed{\frac{1}{2}h}}{\tan(\frac{h}{2})}$$

$$\Rightarrow 2 \frac{T(\theta \boxed{-h}) - T(\theta)}{\boxed{-h}} \cdot \frac{(\boxed{\frac{h}{2}})}{\tan(\boxed{\frac{h}{2}})} \longmapsto 2T'(\theta) \cdot 1$$

Thus we get the differential equation

$$2T'(\theta) = \mu\, 2T(\theta)$$

$$T'(\theta) = \mu\, T(\theta)$$

What functions are essentially unchanged by derivative?

$$T(\theta) = A e^{\mu\theta}$$

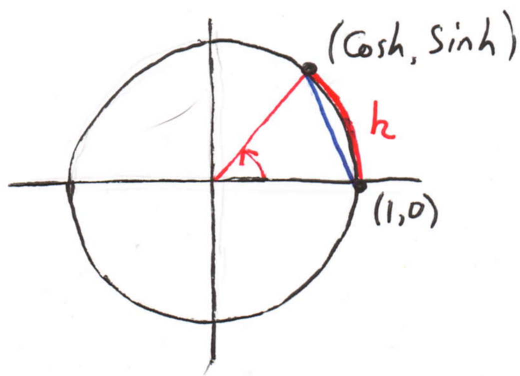
and $B = T(\varphi) = A e^{\mu\varphi}$!

The capstan increases your force exponentially!

Trigonometric Limits

Thm: $\displaystyle \lim_{h \to 0} \frac{\cos h - 1}{h} = 0$

Proof:



By the distance formula

$$0 \le \sqrt{(\cos h - 1)^2 + (\sin h - 0)^2} \le |h|$$

$$0 \le (\cos h - 1)^2 + \sin^2 h \le h^2$$

$$0 \le \underline{\cos^2 h} - 2\cos h + 1 + \underline{\sin^2 h} \le h^2$$

$$0 \le 2 - 2\cos h \le h^2$$

If h is positive, then on dividing by $2h$ we obtain

$$\begin{array}{ccccc} 0 & \le & \dfrac{1-\cos h}{h} & \le & \frac{1}{2}h \\ \downarrow & & \Downarrow & & \downarrow \\ 0 & & 0 & & 0 \end{array}$$

as $h \mapsto 0^+$ by the squeeze thm.

If h is negative, then on dividing by $2h$ we obtain

$$\begin{array}{ccccc} 0 & \ge & \dfrac{1-\cos h}{h} & \ge & \frac{1}{2}h \\ \downarrow & & \Downarrow & & \downarrow \\ 0 & & 0 & & 0 \end{array}$$

as $h \mapsto 0^-$ by the squeeze thm.

Hence $\lim_{h \to 0} \frac{\cos h - 1}{h} = \lim_{h \to 0} - \boxed{\frac{1 - \cos h}{h}} = -1 \cdot \boxed{0} = 0$

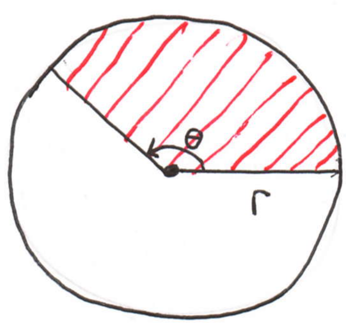
<u>Corollary:</u> $\lim_{h \to 0} \cos h = 1$.

we are trying to make it look like

<u>Proof:</u> $\lim_{h \to 0} \cos h = \lim_{h \to 0} 1 + \boxed{\cos h - 1} =$

$= \lim_{h \to 0} 1 + \boxed{\frac{\cos h - 1}{h}} \cdot h = 1 + \boxed{0} \cdot 0 = 1$.

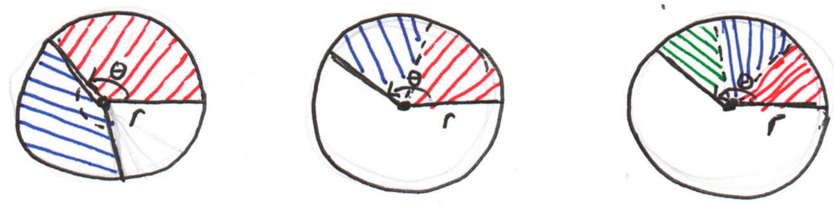
Before proceeding further, we must develop the formula for the area of a circular sector. Consider a sector with radius r and central angle of θ radians.



Think of θ as the central angle of a slice of pizza.

<u>Q.</u> What happens to the area of our slice when we double the angle? Half the angle? Trisect the angle?

A.



$\theta \to 2\theta$ Area doubles | $\theta \to \frac{1}{2}\theta$ Area halves | $\theta \to \frac{1}{3}\theta$ Area reduced to $\frac{1}{3}$.

Thus $A(\theta) \sim \theta$ (the area of the sector is proportional to θ).

If $\theta = 2\pi$, the sector is the full disc. Hence $A(2\pi) = \pi r^2$

By proportionality $A(\theta) = k\,\theta$

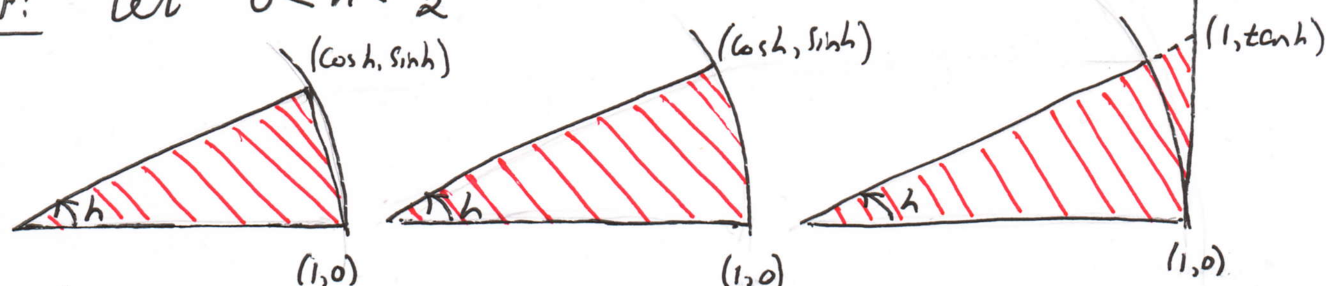
we have $\dfrac{A(\theta)}{\theta} = k$ for all $\theta > 0$.

Thus $k = \dfrac{A(\theta)}{\theta} = \dfrac{A(2\pi)}{2\pi} = \dfrac{\pi r^2}{2\pi} = \dfrac{1}{2} r^2$

In particular, $\boxed{A(\theta) = \frac{1}{2} r^2 \theta}$

Thm: $\displaystyle\lim_{h \to 0} \frac{\sin h}{h} = 1$.

Proof: let $0 < h < \frac{\pi}{2}$



$A = \frac{1}{2}\sin h$ $\qquad$ $A = \frac{1}{2} h$ $\qquad$ $A = \frac{1}{2}\tan h$

Observe that

$$\frac{1}{2}\sin h \leqslant \frac{1}{2}h \leqslant \frac{1}{2}\tan h$$

Upon dividing by $\frac{1}{2}\sin h$ we obtain

$$1 \leqslant \frac{h}{\sin h} \leqslant \frac{\tan h}{\sin h}$$

$$\begin{array}{ccccc} 1 & \leqslant & \dfrac{h}{\sin h} & \leqslant & \cos h \\ \downarrow & & \Downarrow & & \downarrow \\ 1 & & 1 & & 1 \end{array}$$

as $h \mapsto 0^+$ by the squeeze thm.

It follows that $\lim\limits_{h \mapsto 0^+} \dfrac{\sin h}{h} = 1$.

Now $\lim\limits_{h \to 0^-} \dfrac{\sin h}{h} = \lim\limits_{h \to 0^-} \dfrac{-\sin(-h)}{h}$

$= \lim\limits_{h \to 0^-} \dfrac{\sin(-h)}{(-h)} = \lim\limits_{v \to 0^+} \dfrac{\sin(v)}{v} = 1.$