Trigonometry Lecture 3

Here we explore what we can do with the knowledge $\boxed{\lim_{h \mapsto 0} \frac{\cos h - 1}{h} = 0}$ and $\boxed{\lim_{h \mapsto 0} \frac{\sin h}{h} = 1}$.

Ex. Find $\lim_{h \mapsto 0} \frac{\sin 5h}{h}$

Solution: If $\lim \square \mapsto 0$ $\quad \frac{\sin \square}{\square} \mapsto 1$.

So $\lim_{h \mapsto 0} 5 \cdot \frac{\sin \boxed{5h}}{\boxed{5h}} = 5 \cdot 1 = 5$.

Ex. $\lim_{h \mapsto 0} \frac{\sin 6h}{3h} = \lim_{h \mapsto 0} \frac{6}{3} \cdot \frac{\sin \boxed{6h}}{\boxed{6h}} = 2 \cdot 1$.

Ex. $\lim_{h \mapsto 0} \frac{\sin h^2}{h} = \lim_{h \mapsto 0} h \cdot \frac{\sin \boxed{h^2}}{\boxed{h^2}} = 0 \cdot 1$.

Ex. $\lim_{h \mapsto 0} \frac{\sin h}{h^2} = \lim_{h \mapsto 0} \frac{1}{h} \frac{\sin \boxed{h}}{\boxed{h}} = \pm\infty \cdot 1$

limit does not exist.

Ex. $\lim_{x \mapsto \infty} x \sin\left(\frac{1}{x}\right) = \lim_{x \mapsto \infty} \frac{\sin \boxed{\frac{1}{x}}}{\boxed{\frac{1}{x}}} = 1$.

Ex. $$\lim_{t \to 0} \frac{\cos t - 1}{\sin t} = \lim_{t \to 0} \frac{\cos t - 1}{\boxed{t}} \cdot \frac{\boxed{t}}{\sin t}$$

$$= 0 \cdot 1 = 0.$$

Ex. $$\lim_{x \to 1} \frac{\sin(x^2 - 2x + 1)}{(x-1)} =$$

$$= \lim_{x \to 1} \frac{\sin \boxed{(x-1)^2}}{\boxed{(x-1)}} = \lim_{x \to 1} \frac{\sin \boxed{(x-1)^2}}{\boxed{(x-1)^2}} \cdot \boxed{\frac{(x-1)}{1}}$$

$$= \lim_{x \to 1} \frac{\sin (x-1)^2}{(x-1)^2} \cdot (x-1) = 1 \cdot 0 = 0.$$

Ex. $$\lim_{\theta \to 0} \frac{\sin 6\theta}{\sin 3\theta}$$

Solution: (a) $$\lim_{\theta \to 0} \frac{\sin 6\theta}{\sin 3\theta} = \lim_{\theta \to 0} 6 \cdot \frac{\sin \boxed{6\theta}}{\boxed{6\theta}} \cdot \frac{\boxed{3\theta}}{\sin \boxed{3\theta}} \; \frac{1}{3}$$

$$= 6 \cdot \frac{1}{3} = \frac{6}{3} = 2$$

(b) Lets observe what $\lim_{x \to 0} \frac{\sin x}{x} = 1$ really means: If $\boxed{x \text{ is in radians}}$ and small then $\frac{\sin x}{x} \approx 1$. In particular $\sin x \approx x$.

So $$\lim_{\theta \to 0} \frac{\sin 6\theta \approx 6\theta}{\sin 3\theta \approx 3\theta} = \lim_{\theta \to 0} \frac{6\theta}{3\theta} = 2.$$

(c) Some students solve this problem as follows:

$$\lim_{\theta \to 0} \frac{\cancel{\sin}\,\overset{2}{\cancel{6}}\cancel{\theta}}{\cancel{\sin}\,\cancel{3}\cancel{\theta}} = 2.$$

The answer is indeed 2, but the solution reminds me of a remedy against old age:

Push an old grandma off the stairs, as she rolls down the steps she turns into a young beautiful woman.

<u>Ex.</u> $\displaystyle \lim_{t \to 0} \frac{\cos t - 1}{\sin t}$

(a) $\displaystyle \lim_{t \to 0} \frac{\cos t - 1}{t} \cdot \frac{t}{\sin t} = 0 \cdot 1 = 0$

(b) $\displaystyle \lim_{t \to 0} \frac{\cos t - 1}{\sin t} \overset{*}{=} \lim_{t \to 0} \frac{\cos t - 1}{t}$ bcz $\sin t \approx t$ when t is small.

<u>Ex.</u> $\displaystyle \lim_{x \to 0} \frac{\sin ax}{\sin bx} = ?$

$\displaystyle \lim_{x \to 0} \frac{\tan x}{x} = ?$

$\displaystyle \lim_{x \to \infty} x \sin\left(\frac{5}{x}\right) = ?$

Ex. $\lim_{\theta \to 0} \dfrac{\sin 5\theta \ \sin 2\theta}{\theta^2} \overset{*}{=} \lim_{\theta \to 0} \dfrac{5\theta \cdot 2\theta}{\theta^2}$

$= 10.$

Ex. $\lim_{\theta \to 0} \dfrac{\sin 2\theta \cdot \tan 7\theta}{7\theta^2} = ?$

Ex. $\lim_{x \to 1} \dfrac{\sin(x-1)}{x^2+x-2} = ?$

Ex. $\lim_{x \to 0} \dfrac{\sin 3x}{5x^3 - 4x} = ?$

Ex. $\lim_{x \to \frac{\pi}{4}} \dfrac{1-\tan x}{\sin x - \cos x} = \lim_{x \to \frac{\pi}{4}} \dfrac{1 - \frac{\sin x}{\cos x}}{\sin x - \cos x}$

$= \lim_{x \to \frac{\pi}{4}} \underbrace{\boxed{\dfrac{1}{\cos x}}}_{\sqrt{2}} \ \underbrace{\boxed{\dfrac{\cos x - \sin x}{\sin x - \cos x}}}_{-1} = -\sqrt{2}.$

Ex. $\lim_{\theta \to 0} (1+\sin\theta)^{\frac{1}{\theta}} = ?$

Solution: Hint: what does $(1+\square)^{\frac{1}{\square}}$ remind you of?

$$\lim_{\theta \to 0} \left[(1+\sin\theta)^{\frac{1}{\sin\theta}} \right]^{\frac{\sin\theta}{\theta}} = e^{1}.$$

<u>Ex.</u> $\displaystyle \lim_{x \to 0} \frac{e^{\sin x} - 1}{x} = ?$

<u>Ex.</u> $\displaystyle \lim_{\theta \to 0} (1+\sin 5\theta)^{\frac{1}{10\theta}} = \lim_{\theta \to 0} \underbrace{\left[(1+\sin 5\theta)^{\frac{1}{\sin 5\theta}} \right]}_{e}{}^{\overbrace{\frac{\sin 5\theta}{10\theta}}^{}_{\frac{1}{2}}}$

$= e^{\frac{1}{2}}.$

<u>Ex.</u> $\displaystyle \lim_{v \to 0} (1 - \tan(6v))^{\frac{1}{\sin(2v)}} =$

$\displaystyle = \lim_{v \to 0} \underbrace{\left[(1 - \tan(6v))^{\frac{1}{\tan 6v}} \right]}_{e^{-1}}{}^{\underbrace{\frac{\tan 6v}{\sin 2v}}_{3}} = e^{-3}.$

<u>Ex.</u> $\displaystyle \lim_{x \to \infty} x \tan(4x) = ?$

<u>Ex.</u> $\displaystyle \lim_{t \to 0} (1+\sin 5t)^{\frac{1}{\tan 25t}} = ?$

<u>Ex.</u> $\displaystyle \lim_{y \to 0} \frac{\tan\left(\frac{\pi}{4} + \sin y\right) - 1}{y} = ?$

<u>Ex.</u> $\displaystyle \lim_{z \to \frac{\pi}{2}} \frac{\cos z}{z - \frac{\pi}{2}} = ?$ <u>Ex.</u> $\displaystyle \lim_{h \to 0} \frac{\sin h^{\circ}}{h}$ (degrees!)

The Squeeze theorem

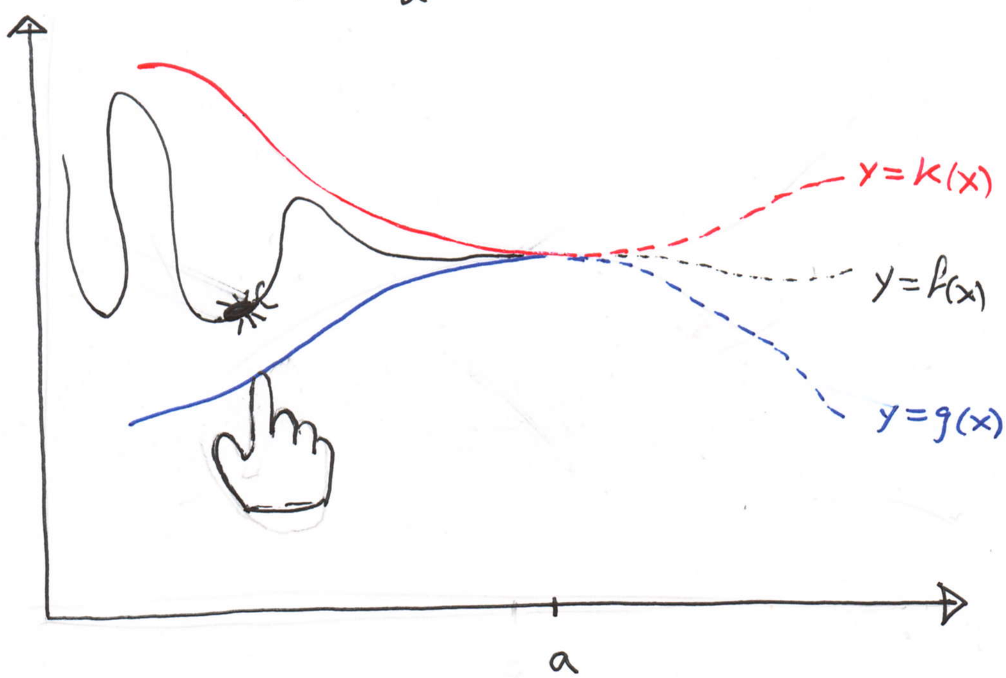
Suppose $\lim_{x \to a} f(x)$ is a difficult limit.

(e.g. a cockroach that doesn't want to be trapped)

If we can find two functions $g(x) \leq f(x) \leq k(x)$

whose limit is $\lim_{x \to a} g(x) = \lim_{x \to a} k(x) = L$

(e.g. the hands that trap the cockroach) then it also follows that $\lim_{x \to a} f(x) = L$.



Ex. Find $\lim\limits_{x \to 0} x^2 \sin\left(\frac{1}{x}\right)$.

Solution:

$$-1 \leq \sin\left(\frac{1}{x}\right) \leq 1$$

$$\begin{array}{cccccc} & -x^2 & \leq & x^2 \sin\left(\frac{1}{x}\right) & \leq & x^2 \\ \text{as } x \to 0 & \downarrow & & \Downarrow & & \downarrow \\ & 0 & & 0 & & 0 \end{array}$$

Ex. Find $\lim\limits_{x \to 2} 2(x-2)^4 \sin\left(\frac{1}{x-2}\right)\cos\left(\frac{1}{x-2}\right)$

Solution: (a) Observe that $2\sin\left(\frac{1}{x-2}\right)\cos\left(\frac{1}{x-2}\right)$

$= \sin\left(\frac{2}{x-2}\right)$

we want to find "nice" functions that bound the limit expression from above and below.

$$-1 \leq \sin\left(\frac{2}{x-2}\right) \leq 1$$

Hence $-(x-2)^4 \leq (x-2)^4 \sin\left(\frac{2}{x-2}\right) \leq (x-2)^4$

$$\begin{array}{ccc} \downarrow & \Downarrow & \downarrow \\ 0 & 0 & 0 \end{array}$$

as $x \to 2$.

(b) Alternatively, observe that

$$-1 \leq \sin\left(\frac{1}{x-2}\right)\cos\left(\frac{1}{x-2}\right) \leq 1$$

Hence, upon multiplying by $2(x-2)^4$

$$\underset{\substack{\downarrow \\ 0}}{-2(x-2)^4} \leq \underset{\substack{\Downarrow \\ 0}}{2\sin\left(\frac{1}{x-2}\right)\cos\left(\frac{1}{x-2}\right)(x-2)^4} \leq \underset{\substack{\downarrow \\ 0}}{2(x-2)^4}$$

as $x \to 2$.

Ex. $\displaystyle\lim_{x \to 1} \underbrace{10(x-1)^2 e^x \tan\left(\tfrac{\pi}{4}x\right)}_{g(x)} \cos\left(\frac{1}{x-1}\right) + 5$

Solution: $\quad -10 \leq 10\cos\left(\frac{1}{x-1}\right) \leq 10$

$$\underset{\substack{\downarrow \\ 0}}{-10\underbrace{(x-1)^2 e^x \tan\left(\tfrac{\pi}{4}x\right)}_{g(x)}} \leq \underset{\substack{\Downarrow \\ 0}}{10(x-1)^2 e^x \tan\left(\tfrac{\pi}{4}x\right)\cos\left(\frac{1}{x-1}\right)} \leq \underset{\substack{\downarrow \\ 0}}{10\, g(x)}$$

Thus the limit is 5.