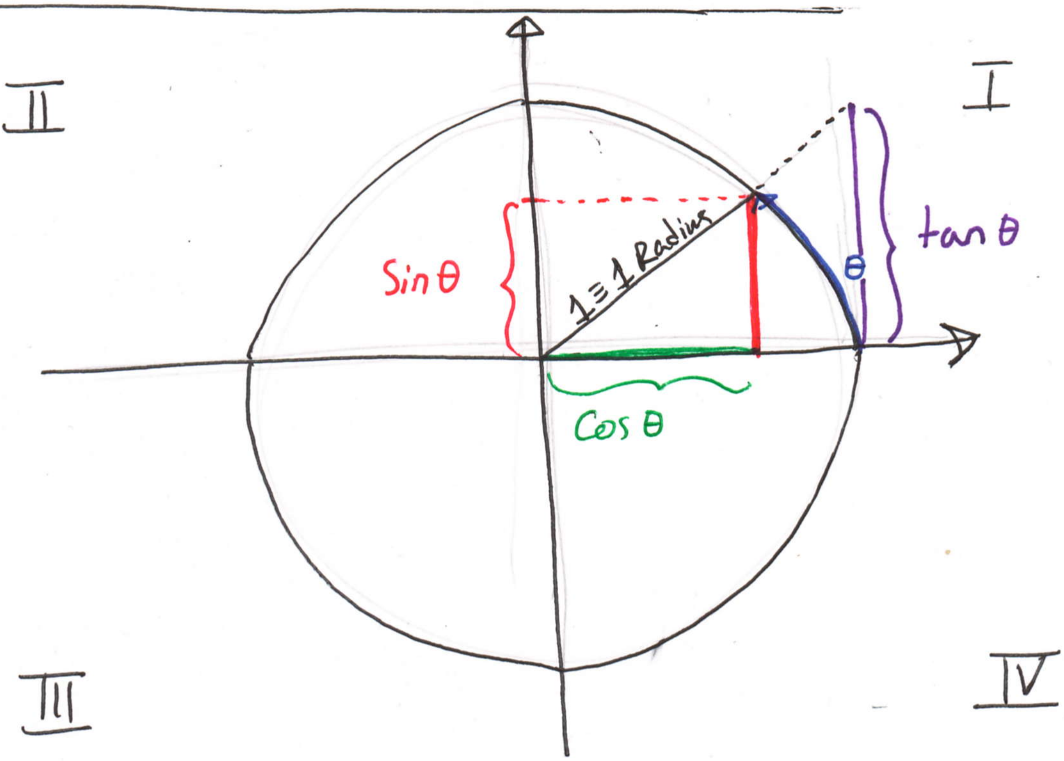
Trigonometry Lecture Notes 2



θ measured in radians (length of pizza crust in radius units)

- positive θ indicates counter clockwise movement.
- negative θ indicates clockwise movement.

$\sin\theta$ measures the length of line segment indicated in red in radius units. (e.g. $\sin\theta = 0.9$ means 90% of radius)

- $\sin\theta > 0$ if red line segment is measured against positive y-axis.
- $\sin\theta < 0$ if line segment is measured against negative y-axis.
- $\sin\theta \neq \sin\cdot\theta$ this would have been meaningless! Rather this is a command: (1) Open angle to θ (2) Drop $\perp$ line (3) measure the length of line segment in radius units.

- $\sin\theta$ doesn't explicitly specify algebraic manipulations (i.e. $+ \times, \%$) that lead to a computation in terms of θ.

$\cos\theta$ is to be understood similarly.

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\text{Vertical segment}}{\text{Horizontal segment}}$$

- by extending horizontal segment to 1 we get a larger similar triangle. Hence $\frac{\text{length}(|)}{\text{length}(-)} = \frac{\text{length}(|)}{1}$ $= \tan\theta$

Questions to consider

1) Can we find a polynomial formula that will compute the trig function in terms of θ?

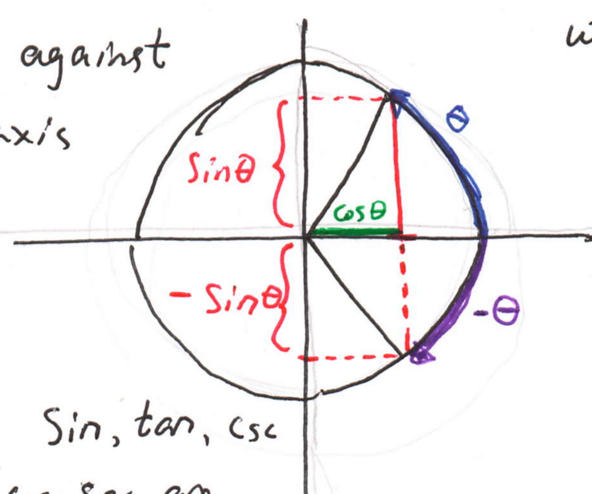
2) Why radian measure is natural and degree measure isn't?

Important properties for trig functions

1) Sin is odd and Cos is even.

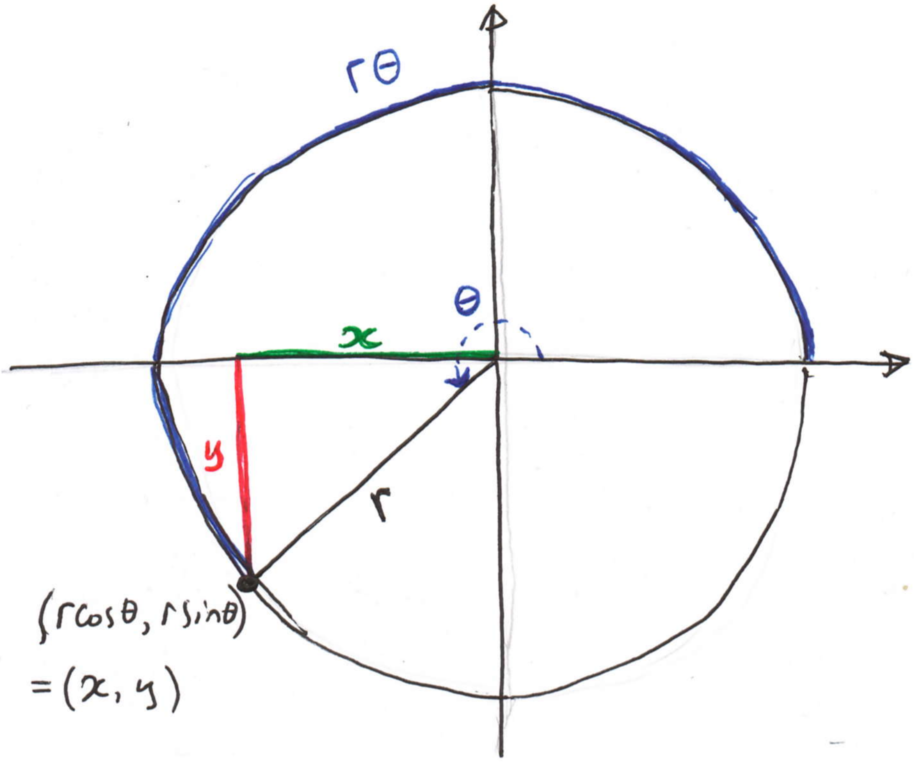
Cos remains unchanged when θ is replaced by $-\theta$.

$\sin(-\theta)$ measures against the negative y-axis Hence $\sin(-\theta)$ $= -\sin(\theta)$



It follows that sin, tan, csc cot are odd. and cos, sec are even. (why?)

2) The x and y coordinates can be related as follows:



$$\frac{x}{r} = \cos\theta \Rightarrow x = r\cos\theta \qquad \frac{y}{r} = \sin\theta \Rightarrow y = r\sin\theta$$

Questions

(a) x is colored green, shouldn't it simply be $x = \cos\theta$?

(b) Similarly y is $\perp$ red. shouldn't y simply be $y = \sin\theta$?

(c) The blue arc is labled "$r\theta$", why not "θ"?

Think about it intensely before looking up what I wrote.

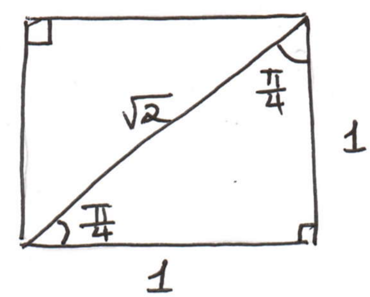
Answers

(a)-(c) The scale isn't in radius units. $\frac{y}{r}$ for instance re-scales from the given units to a measurement in terms of r. $\left(\frac{y}{r} \times 100\right)\% \equiv$ portion of y in terms of r.

3) By Pythagoras' thm $\cos^2\theta + \sin^2\theta = 1$. Upon dividing equation by $\cos^2\theta$, we obtain $1 + \frac{\sin^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta}$ or $1 + \tan^2\theta = \sec^2\theta$. Similarly, we may obtain $\cot^2\theta + 1 = \csc^2\theta$

Make sure you see this!!!

4) Most trig values are difficult to compute exactly. Here are several exceptions:



$\sin\left(\frac{\pi}{4}\right) = \sin(45°) =$

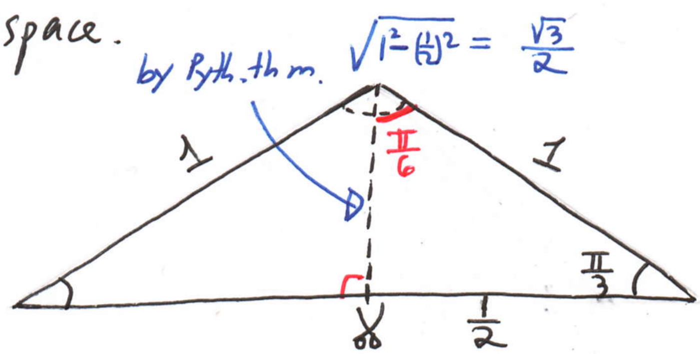
$$= \frac{\text{Opp}\measuredangle}{\text{Hyp.}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

Practice

(a) Calculate cos, tan, sec, csc, cot of $\frac{\pi}{4}$.

(b) What are the values of the six trig. functions at $\frac{3\pi}{4}$? At $\frac{5\pi}{4}$?

Remark: I hate mnemonics (e.g. socahtoa). If you need to memorize this then you don't understand. Memory storage coupled with understanding requires a minimum of storage space.



Equilateral triangle. All angles are equal and their radian value $= \frac{\pi}{3}$

Practice

(a) Compute the values of the six trig. functions at $\frac{\pi}{3}$ and $\frac{\pi}{6}$.

(b) What is the value of $\sin\left(\frac{2\pi}{3}\right)$? Compute the values of all other trig functions at $\frac{2\pi}{3}$!

(c) Repeat (b) for the angle $\frac{5\pi}{6}$.

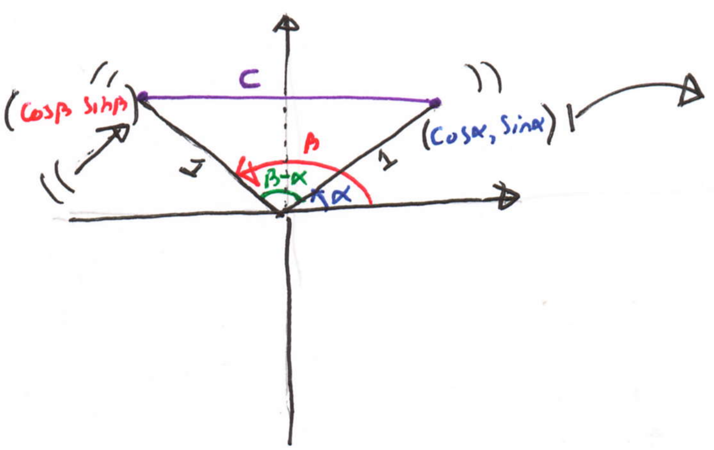
(d) Repeat (b) for the angles $\frac{4\pi}{3}$, $\frac{5\pi}{3}$, $\frac{7\pi}{6}$, $\frac{11\pi}{6}$.

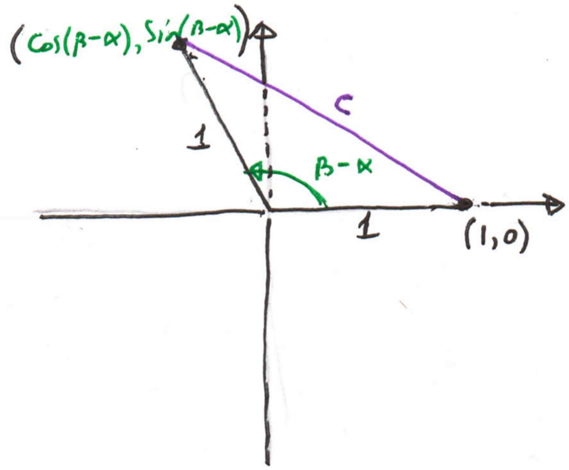
5) Sum and difference of angles. This is very important!! Make sure to study thoroughly.

We wish to calculate $\cos(\beta-\alpha)$ in terms of $\sin\alpha$, $\cos\alpha$, $\sin\beta$, $\cos\beta$. A moment's thought should convince you that

$$\cancel{\cos(\beta-\alpha) = \cos(\beta)-\cos(\alpha)}.$$ For instance

$1 = \cos(0-0) \neq \cos(0)-\cos(0) = 1-1 = 0$. Don't step on mines and then pray to God the mine doesn't explode!





- Picture illustrates the same triangle in different positions. The left triangle is set with one leg at an angle α and the other leg at an angle β with repect to the x-axis. The rigth triangle results when we rotate the left triangle until its left leg is flat on the x-axis.
- The angle $\beta-\alpha$ is indicated in both pictures.
- Do you see how the coordinates were labled in terms of Sin and Cos? Why? Refer to the picture on page 3. If you still don't understand move to the next bullet point. Come back if confusion persists.
- It follows from the distance formula that

$$c^2 = (\cos\beta - \cos\alpha)^2 + (\sin\beta - \sin\alpha)^2 =$$

$$= \underline{\cos^2\beta} - 2\cos\beta\cos\alpha + \underline{\cos^2\alpha} + \underline{\sin^2\beta} - 2\sin\beta\sin\alpha + \underline{\sin^2\alpha}$$

$$= 2 - 2(\cos\beta\cos\alpha + \sin\beta\sin\alpha)$$

where we recall that $\cos^2 \cdot + \sin^2 \cdot = 1$.

On the other hand, computing c^2 from the figure on the right results in

$$c^2 = \left\|(\cos(\beta-\alpha), \sin(\beta-\alpha)) - (1,0)\right\|^2 = (\cos(\beta-\alpha)-1)^2 + \sin^2(\beta-\alpha)$$

$$= \underline{\cos^2(\beta-\alpha)} - 2\cos(\beta-\alpha) + \underline{1} + \underline{\sin^2(\beta-\alpha)}$$

$$= 2 - 2\cos(\beta-\alpha)$$

Equating the two formulas for c^2, we obtain

$$2 - 2(\cos\beta\cos\alpha + \sin\beta\sin\alpha) = 2 - 2\cos(\beta-\alpha)$$

Solving for $\cos(\beta-\alpha)$, we obtain

$$\boxed{\cos(\beta-\alpha) = \cos\beta\cos\alpha + \sin\beta\sin\alpha}$$

This formula implies that

$$\cos\left(\tfrac{\pi}{2}-\theta\right) = \underbrace{\cos\tfrac{\pi}{2}}_{=0}\cos\theta + \underbrace{\sin\tfrac{\pi}{2}}_{=1}\sin\theta$$

$$= \sin\theta$$

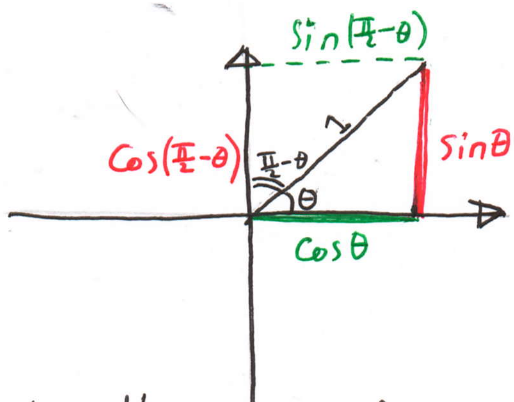
In other words, sin of Any!! angle equals to $\cos\left(\frac{\pi}{2} - \text{that angle}\right)$.

Thus, we also observe

$$\sin\Big(\underbrace{\tfrac{\pi}{2}-\theta}_{\text{angle}}\Big) = \cos\Big(\underbrace{\tfrac{\pi}{2}-\left[\tfrac{\pi}{2}-\theta\right]}_{\frac{\pi}{2}-\text{ that angle}}\Big) = \cos\theta$$

Hence

$$\boxed{\begin{aligned} \cos\left(\tfrac{\pi}{2}-\theta\right) &= \sin\theta \\ \sin\left(\tfrac{\pi}{2}-\theta\right) &= \cos\theta \end{aligned}}$$



This might have been geometrically clear from the picture. We will put these formulas to use in a moment.

First

$$\cos(\beta+\alpha) = \cos(\beta-(-\alpha)) = \cos\beta\,\underbrace{\cos(-\alpha)}_{=\cos\alpha} + \sin\beta\,\underbrace{\sin(-\alpha)}_{=-\sin\alpha}$$

$$= \cos\beta\cos\alpha - \sin\beta\sin\alpha$$

Now

$$\sin(\beta+\alpha) = \cos\left(\frac{\pi}{2}-[\beta+\alpha]\right) = \cos\left(\left[\frac{\pi}{2}-\beta\right]-\alpha\right)$$

$$= \underbrace{\cos\left(\frac{\pi}{2}-\beta\right)}_{=\sin\beta}\cos\alpha + \underbrace{\sin\left(\frac{\pi}{2}-\beta\right)}_{=\cos\beta}\sin\alpha$$

$$= \sin\beta\cos\alpha + \cos\beta\sin\alpha$$

$$\sin(\beta-\alpha) = \sin(\beta+(-\alpha)) = \sin\beta\,\underbrace{\cos(-\alpha)}_{=\cos\alpha} + \cos\beta\,\underbrace{\sin(-\alpha)}_{=-\sin\alpha}$$

$$= \sin\beta\cos\alpha - \cos\beta\sin\alpha$$

Here are the formulas again

$$\boxed{\begin{aligned}
\cos(\beta-\alpha) &= \cos\beta\cos\alpha + \sin\beta\sin\alpha \\
\cos(\beta+\alpha) &= \cos\beta\cos\alpha - \sin\beta\sin\alpha \\
\sin(\beta-\alpha) &= \sin\beta\cos\alpha - \cos\beta\sin\alpha \\
\sin(\beta+\alpha) &= \sin\beta\cos\alpha + \cos\beta\sin\alpha
\end{aligned}}$$

An immediate consequence is

$$\boxed{\begin{aligned} \cos(2\theta) &= \cos(\theta+\theta) = \cos^2\theta - \sin^2\theta \\ &= 1-2\sin^2\theta \\ &= 2\cos^2\theta - 1 \\ \sin(2\theta) &= 2\sin\theta\cos\theta \end{aligned}}$$

Practice

Can you produce a similar formula for $\tan(\beta-\alpha)$, $\tan(\beta+\alpha)$, $\tan(2\theta)$?

Half-angle formulas

Notice that $\cos(\theta) = \cos(2\cdot\frac{\theta}{2}) = 2\cos^2(\frac{\theta}{2}) - 1$.

Thus $\boxed{\cos^2(\frac{\theta}{2}) = \frac{1+\cos(\theta)}{2}}$

Similarly $\cos(\theta) = \cos(2\cdot\frac{\theta}{2}) = 1-2\sin^2(\frac{\theta}{2})$

Thus $\boxed{\sin^2(\frac{\theta}{2}) = \frac{1-\cos(\theta)}{2}}$.

Ex. Compute $\sin(\frac{\pi}{8})$

$$\sin^2(\tfrac{\pi}{8}) = \frac{1-\cos(2\frac{\pi}{8})}{2} = \frac{1-\cos(\frac{\pi}{4})}{2} = \frac{1-\frac{1}{\sqrt{2}}}{2} = \frac{2-\sqrt{2}}{4}$$

Hence $\sin(\frac{\pi}{8}) = \sqrt{\frac{2-\sqrt{2}}{4}} = \frac{\sqrt{2-\sqrt{2}}}{2}$

Practice

Find exact value of $\sin(\frac{\pi}{12})$, $\cos(\frac{\pi}{12})$, $\tan(\frac{\pi}{12})$.

Finding the derivatives of trig functions

$$\frac{d}{dx}\sin x = \lim_{h \to 0} \frac{\sin(x+h) - \sin(x)}{h}$$

$$= \lim_{h \to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h}$$

$$= \lim_{h \to 0} \sin x \frac{\cos h - 1}{h} + \cos x \frac{\sin h}{h}$$

looks like $\left.\frac{d}{dx}\cos x\right|_{x=0}$ looks like $\left.\frac{d}{dx}\sin x\right|_{x=0}$

As we shall see, $\lim_{h \to 0} \frac{\cos h - 1}{h} = 0$ and $\lim_{h \to 0} \frac{\sin h}{h} = 1$.

These are not simple limits. Taking this for granted for a moment, it follows that

$$\boxed{\frac{d}{dx}\sin x = \cos x}$$

Practice

Assuming $\lim_{h \to 0} \frac{\cos h - 1}{h} = 0$ and $\lim_{h \to 0} \frac{\sin h}{h} = 1$

Show that $\boxed{\frac{d}{dx}\cos x = -\sin x}$

Having established that $\frac{d}{dx}\sin x = \cos x$ and $\frac{d}{dx}\cos x = -\sin x$ we get the derivatives of other functions with ease.

$$\frac{d}{dx}\tan x = \frac{d}{dx}\frac{\sin x}{\cos x} \overset{\text{Quotient Rule}}{=\!=} \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$

$$= \frac{1}{\cos^2 x} = \sec^2 x.$$

$$\frac{d}{dx}\sec x = \frac{d}{dx}\frac{1}{\cos x} = \frac{\cos x \cdot 0 - (-\sin x)\cdot 1}{\cos^2 x} = \frac{\sin x}{\cos^2 x}$$

$$= \frac{\sin x}{\cos x}\cdot\frac{1}{\cos x} = \tan x \sec x$$

Practice

Establish the formulas for $\frac{d}{dx}\csc x$, $\frac{d}{dx}\cot x$.

Ex. $\frac{d}{dx}(x^2\tan x) = 2x\tan x + x^2\sec^2 x$ by product rule.

$$\frac{d}{dx}\left(\frac{\sec x}{1+x^2}\right) = \frac{(1+x^2)\sec x\tan x - 2x\sec x}{(1+x^2)^2}$$ by quotient rule.

$$\frac{d}{dx}e^x\cos x = e^x\cos x - e^x\sin x$$ by product rule.

Practice

1) $\frac{d}{dx}(xe^x\tan x)$

2) $\frac{d}{dx}(7x^2 - \sec x + \sec x\csc x)$

3) $\frac{d}{dx}\left(\frac{\cos x\tan x}{e^x+5x^2}\right)$