Trigonometry Lecture 1

Degree and Radian Measure

Q. How many degrees are in a circle? Why?

A. The circle has 360°. The reason for this is arbitrary. The Babylonians were fond of the number 60 and used it as the base of their number system. By contrast, our number system has 10 as its base. Perhaps this is due to the 10 fingers on our hands. The base x system represents integers and other numbers as a polynomial of the form

$a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$, where each $a_k \in \{0, 1, \dots, x-1\}$

For example $153 = 1 \cdot 10^2 + 5 \cdot 10^1 + 3 \cdot 10^0$ and

$33.533 = 3 \cdot 10^1 + 3 \cdot 10^0 + 5 \cdot 10^{-1} + 3 \cdot 10^{-2} + 3 \cdot 10^{-3}$

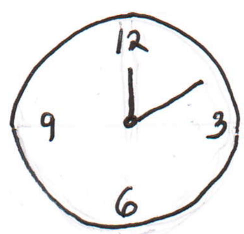
$153 \text{ (base 9)} = 1 \cdot 9^2 + 5 \cdot 9^1 + 3 \cdot 9^0 = 129 \text{ (base 10)}$

$153 \text{ (base 6)} = 1 \cdot 6^2 + 5 \cdot 6^1 + 3 \cdot 6^0 = 69 \text{ (base 10)}$

360 (base 10) $= 6\cdot 60^1 + 0\cdot 60^0 = 60$ (base 60)

From the Babylonians we have retained the description of time: 1 hour = 60 minutes, 1 minute = = 60 seconds etc.

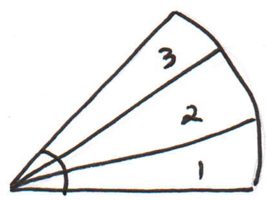
Do you remember that mechanical clocks are circular?



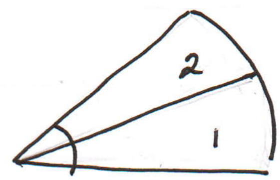
Q. What do the degrees measure and how do they do it?

A. Degrees measure angles. You may think of an angle as a measurement of the sharpness of a corner or wedge.
Imagine cutting a pizza pie into 360 equal slices (instead of the usual 8). 1° is one slice 2° are two slices, etc.

A wedge that can fit in more slices is less sharp, while a wedge that can fit fewer slices is sharper.



This wedge is $3^\circ \equiv 3$ slices sharp.



This wedge is $2^\circ \equiv 2$ slices sharp

Degrees are not "natural" as units of measuring angles, because there is no logically compelling reason to divide the "pizza" into 360 slices any more than there is to cut it into 8.

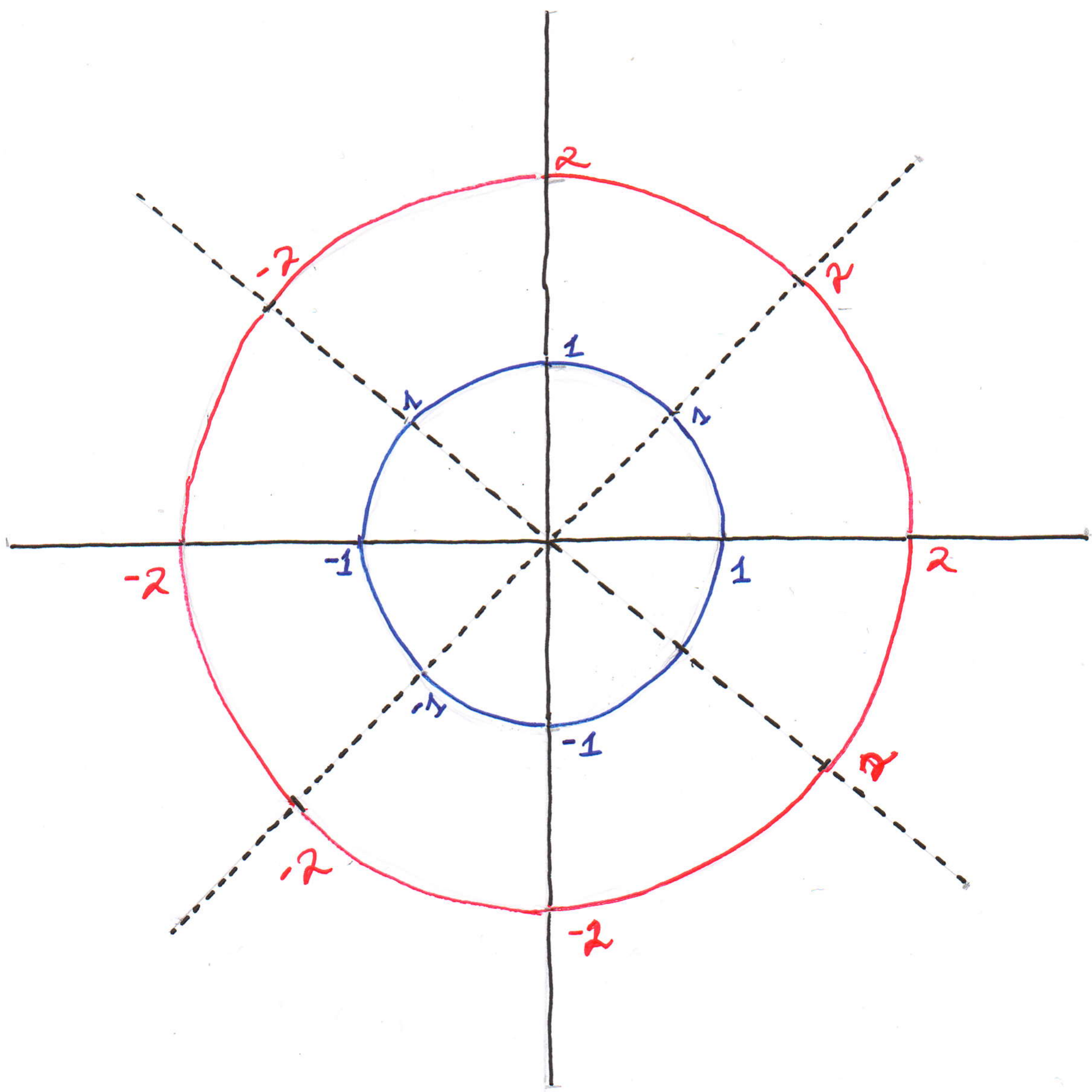
Q. Is there a "natural" measure for angles?

A. A natural measurement of an angle would be one that conveys the sharpness of a circular sector in terms of a property inherent to the circle. Such a measure was discovered by Archimedes.

It is called radian measure.

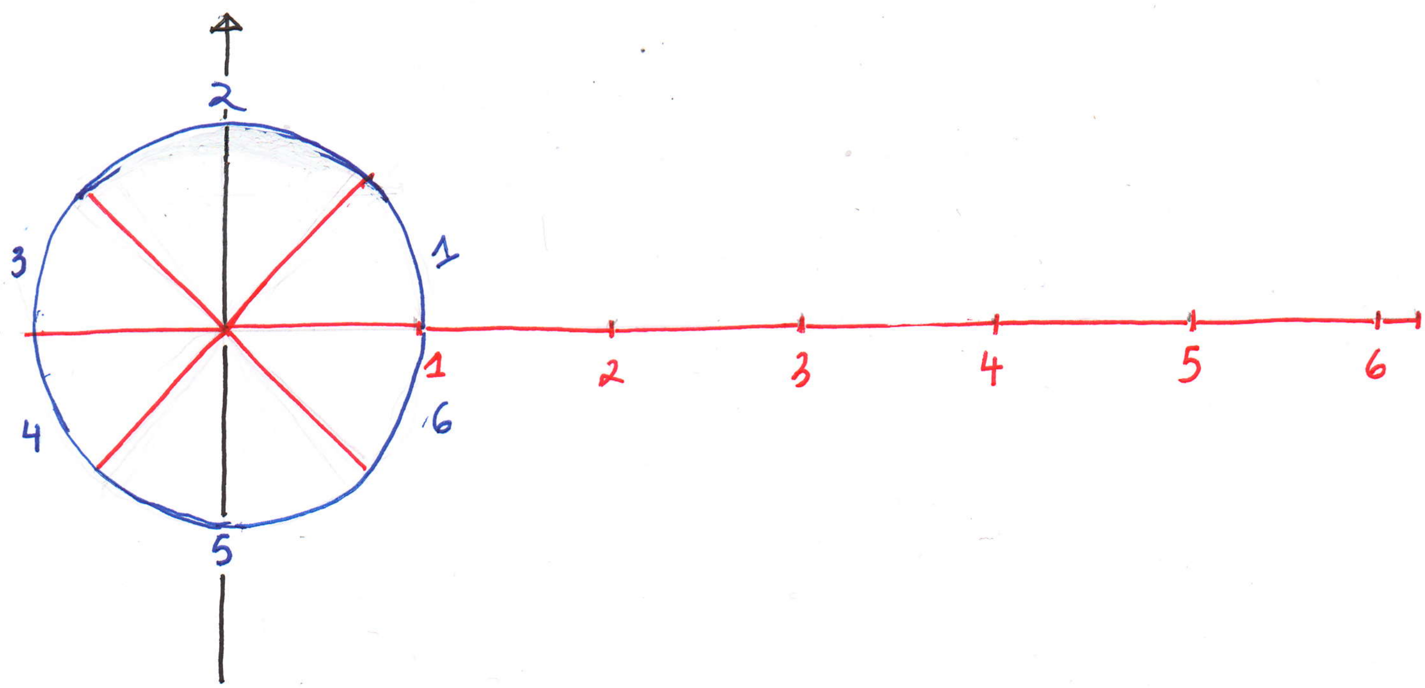
Archimedes discovered the famous relationship between the radius of a circle and its circumference

$C = 2\pi r$.



Since $2\pi \approx 6.28$, the identity $C = 2\pi r$ states that the circumference of a circle is approximately 6.28 radii long.

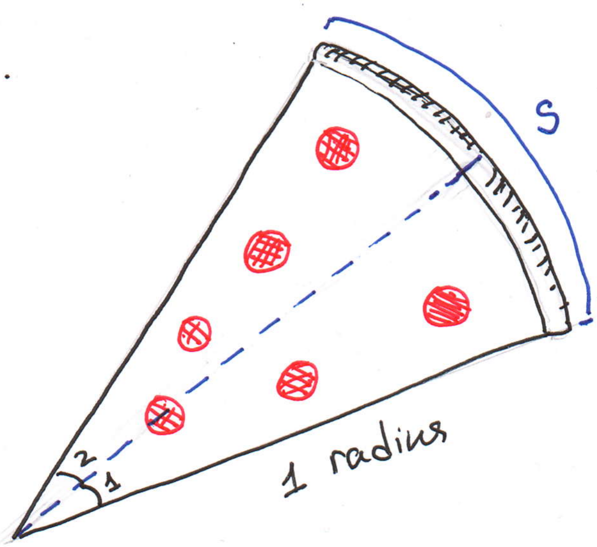
Notice that this fact is independent of the units in which we measure the radius. If $r = 1$ in, $C = 2\pi$ in ≈ 6.28 in. If $r = 2$ in $C = 2\pi(2\text{ in})$ $= 2\pi(\text{radius length}) \approx 6.28$ copies of radius.



In the figure above the pizza was divided into 6 slices the length of each crust is approximately 1 radius long.

Q. So what is radian measure and how does it differ from degree?

A. Think of an angle as the tip of a pizza slice. Degrees measure this angle by counting the number of the 360 slices that fit within the given slice. Radian measure describes the crust of the slice in radius length.



$2^\circ \equiv$ this slice contains two of the 360 slices

$s \text{ rad} \equiv$ this slice has crust of length s radii.

The Relationship between Radians and Degrees

How many radians does one degree carry?
In other words, what is the length of the crust of one of the 360 slices in units of radius length?

We know that

$$360^\circ \equiv 2\pi \text{ rad}$$

i.e. 360 slices carry the full pizza pie crust.

Since each slice carries an equal amount of crust,

$$1^\circ \equiv \frac{2\pi}{360} \text{ rad} = \frac{\pi}{180} \text{ rad}.$$

Ex. Convert degrees to radians

(a) 2° (b) 180° (c) 120°

(d) 60° (e) 90° (f) 45°

Solution:

(a) $2^\circ = 2 \cdot 1^\circ \equiv 2\frac{\pi}{180} = \frac{\pi}{90}$ rad.

(b) $180^\circ = 180 \cdot 1^\circ \equiv 180\frac{\pi}{180} = \pi$ rad.

(c) $120^\circ = 120 \cdot 1^\circ \equiv 120\frac{\pi}{180} = \frac{2}{3}\pi$ rad

(d) $60^\circ = 60 \cdot 1^\circ \equiv 60\frac{\pi}{180} = \frac{\pi}{3}$ rad.

(e) $90^\circ = 90 \cdot 1^\circ \equiv 90\frac{\pi}{180} = \frac{\pi}{2}$ rad

(f) $45^\circ = \frac{1}{2} \cdot 90^\circ \equiv \frac{1}{2}\frac{\pi}{2} = \frac{\pi}{4}$ rad.

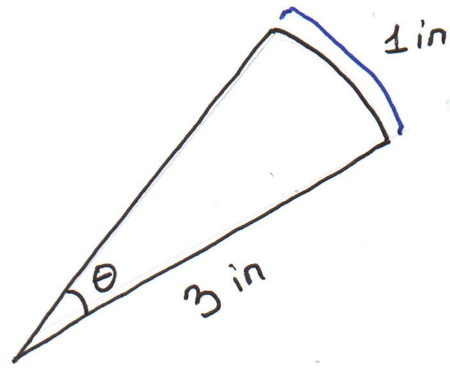
The conversion of radians to degrees is very similar.

$$2\pi \text{ rad} \equiv 360^\circ$$

$$1 \text{ rad} \equiv \frac{360}{2\pi} = \left(\frac{180}{\pi}\right)^\circ$$

In other words, a slice of pizza whose crust is as long as its edge contains $\frac{180^\circ}{\pi} \approx 57.296$ slices from among 360 degree slices.

Ex. How many degrees are in this sector?



Solution:

$1 \text{ rad} \equiv 3 \text{ in}$ so $\frac{1}{3} \text{ rad} \equiv 1 \text{ in}.$

Now $360^\circ \equiv 2\pi \text{ rad}$ so $\left(\frac{180}{\pi}\right)^\circ \equiv 1 \text{ rad}.$

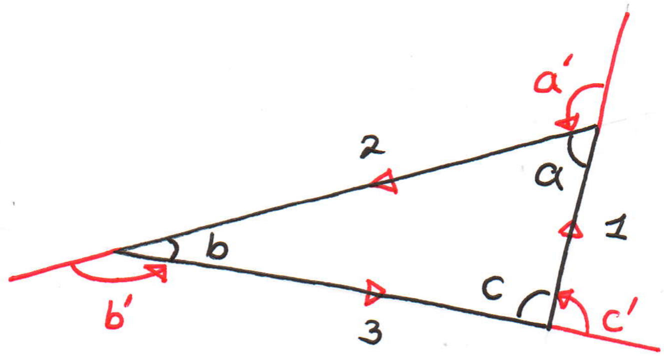
$\frac{1}{3} \text{ rad} \equiv \frac{1}{3} \; \frac{180}{\pi} = \left(\frac{60}{\pi}\right)^\circ \approx 19.099$

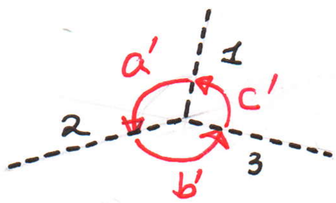
Hence this circular sector contains about 19 of the 360 degree slices.

Polygons and Angles

Q. What is the sum of angles of a triangle?

A.





Observe that as we move around the triangle we rotate a full $360°$ or 2π radians; $a' + b' + c' = 2\pi$

Notice also that $a' + a = b' + b = c' + c = \pi$, because they are supplementary angles (i.e. together they complete a line, which is half of the circle).

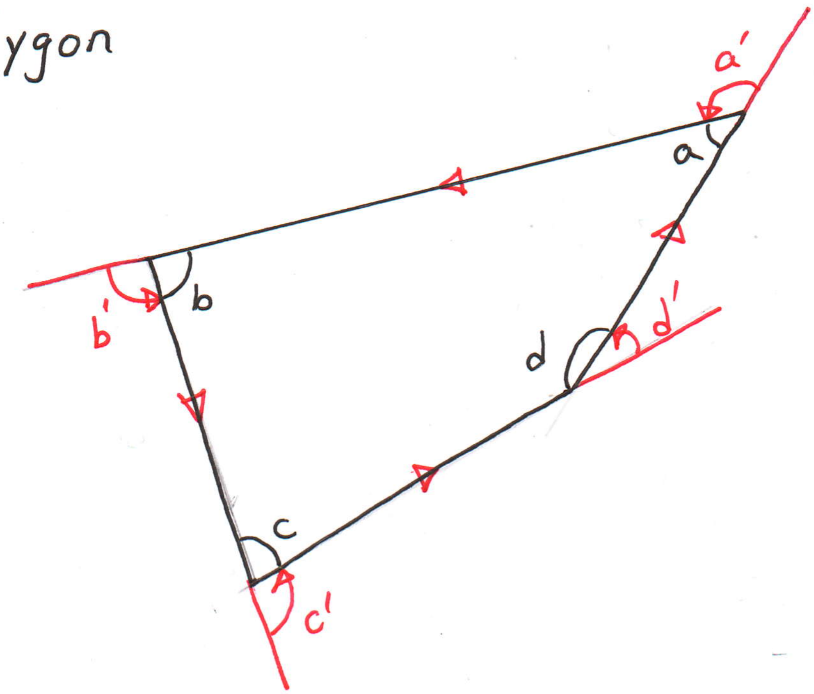
Thus $(a' + a) + (b' + b) + (c' + c) = 3\pi$

$$(a' + b' + c') + (a + b + c) = 3\pi$$

$$2\pi + (a + b + c) = 3\pi$$

$$a + b + c = 3\pi - 2\pi = \pi$$

Ex. Find the sum of the inner angles of the polygon



Solution:

Observe that the outer angles add up to 360° or 2π radians: $a'+b'+c'+d' = 2\pi$

Moreover each $\measuredangle'$ is a supplementary angle to $\measuredangle$.

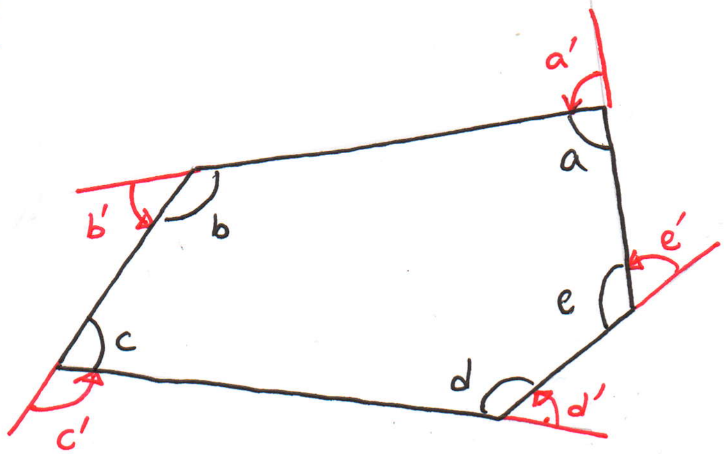
Hence $(a'+a)+(b'+b)+(c'+c)+(d'+d) = 4\pi$

$$\underbrace{(a'+b'+c'+d')}_{2\pi} + (a+b+c+d) = 4\pi$$

$$a+b+c+d = 4\pi - 2\pi = 2\pi.$$

Ex. Find the sum of the angles of any polygon with 5 vertices.

Solution:



Observe that, as before, $\measuredangle' + \measuredangle = \pi$ and

$a' + b' + c' + d' + e' = 2\pi$

Hence $(a'+a) + (b'+b) + (c'+c) + (d'+d) + (e'+e) = 5\pi$

and $a+b+c+d+e = 5\pi - 2\pi = 3\pi$.

Q. What is the sum of angles in a polygon that has n vertices?

A. The sum of the vertex angles $a_1 + a_2 + \ldots + a_n$

$= n\pi - 2\pi = (n-2)\pi$.

Simply observe that the supplementary angles

$a_1' + \ldots + a_n' = 2\pi$

Hence $\underbrace{(a_1' + a_1)}_{\pi} + \underbrace{(a_2' + a_2)}_{\pi} + \ldots + \underbrace{(a_n' + a_n)}_{\pi} = n\pi$

$\Rightarrow \underbrace{(a_1' + a_2' + \ldots + a_n')}_{2\pi} + (a_1 + a_2 + \ldots + a_n) = n\pi$

From, which the desired result follows.

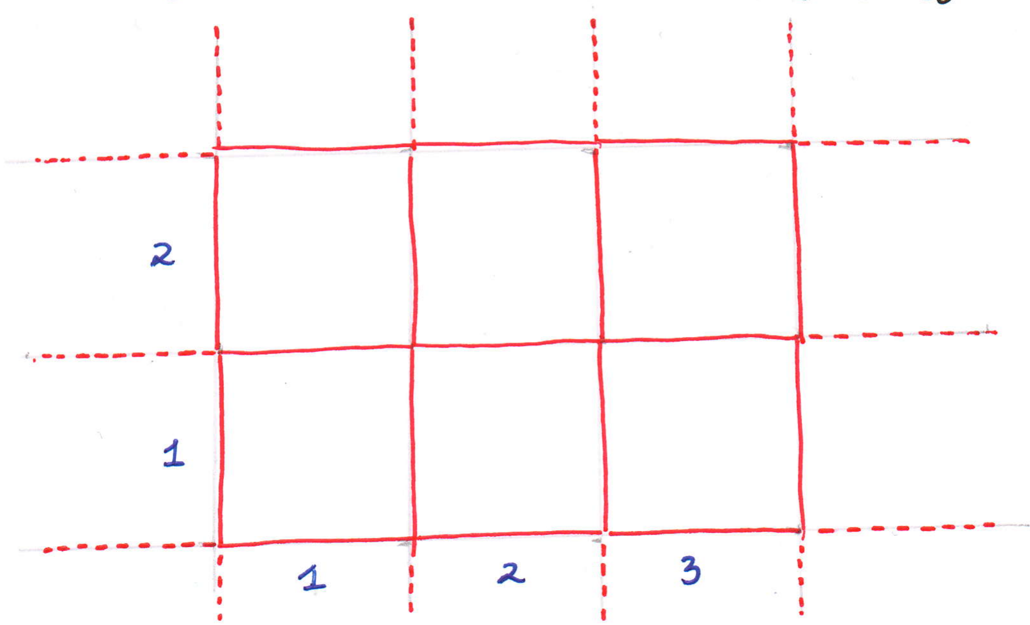
Remark: The statement that the sum of the vertices in any triangle add up to π is a statement about the very nature of geometry. It predicts that our geometry is "flat" or Eucledian geometry. The theory of relativity predicts a different kind of geometry. – one where the angles inside a triangle do not add up to π.

The connection between the sum of angles in a triangle and the very nature of space has already been observed by the great mathematician Gauss.

See Gauss's Theorema Egregium (Latin for "Remarkable Theorem")

Areas and how we communicate the vastness of space

An empty, flat, 2D space can be tessellated with square tiles. The amount of space can then be described as the number of tiles that it contains.

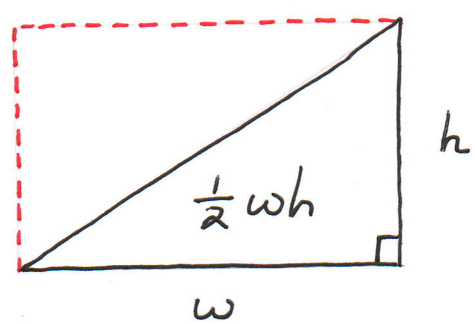


The area of a rectangle is particularly simple to calculate; A rectangle that is 3 squares wide and 2 squares heigh contains 6 square tiles. More generally, a rectangle of height h and width w has area hw. That it contains hw tiles is clear!

<u>Problem:</u> Derive the formula for the area of a triangle.

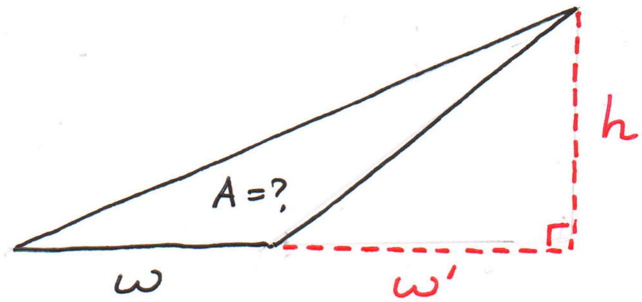
<u>Solution:</u>

First let's figure this area for a right triangle:



The right triangle is half of a rectangle. Hence its area is $\frac{1}{2}$ width $\times$ height of rectangle.

For other triangles we can calculate as follows:



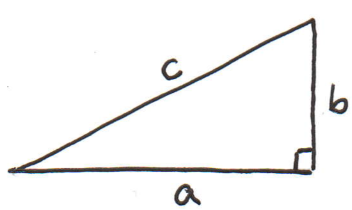
The area of the black triangle is the area of the big right triangle of width $w+w'$ minus the area of the small (red) right triangle of width w'.

Hence $A = \frac{1}{2}(w+w')h - \frac{1}{2}w'h = \frac{1}{2}wh$.

The area of any triangle is $\frac{1}{2}$ width $\times$ height.

The Pythagorean Theorem

Most people remember that the relationship between

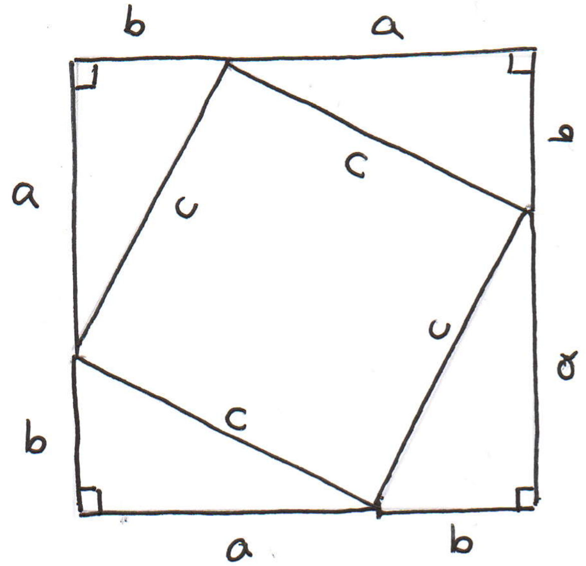


$$a^2+b^2=c^2$$

the legs of the right triangle and its hypotenuse.

But how many of you are able to prove it?

Here is a particularly simple proof:



Make 3 copies of the right triangle and arrange the triangles to form the square in the figure above.

Notice c^2 = Area of square of side c.

$(a+b)^2$ = Area of big square of side $a+b$.

$\frac{1}{2}ab$ = Area of each right triangle.

Now, Area of small square $= c^2 =$ Area of big square minus the areas of triangles.

In particular

$$c^2 = (a+b)^2 - 4 \cdot \tfrac{1}{2} ab = a^2 + \boxed{2ab} + b^2 \boxed{-2ab}$$
$$= a^2 + b^2$$

The Pythagoreans are famous for the saying "All is number". By that, I think, they meant that every segment of space is in proportion to every other segment.

For instance, can you answer these questions:

How tall are you?

How long have you studied?

How much do you weigh?

How fast are you driving?

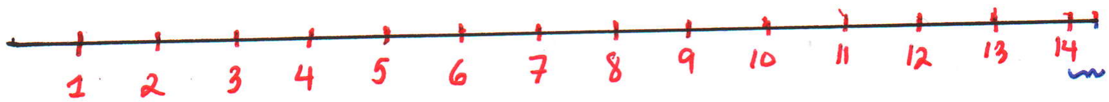
Is the future determined?

Is there such thing as cause and effect?

Is time travel possible?

These questions take their root in the problem of measurement.

How many centimeters is this line segment?



This is approximately 14 cm. Measuring a bit more carefully, we see that the line segment is exactly

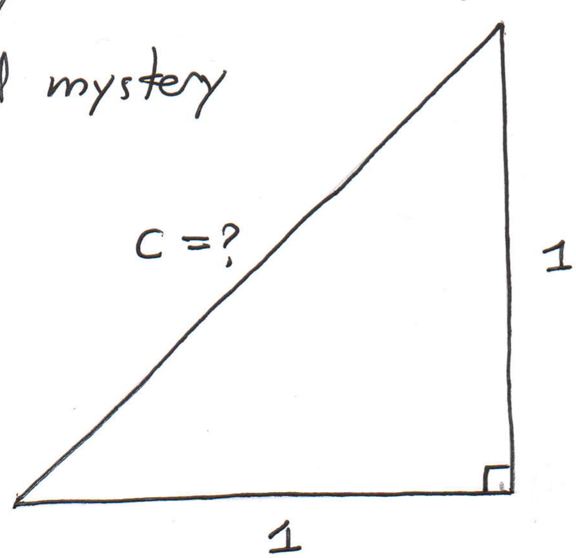
$14 \text{ cm} + \frac{4}{10} \text{ cm}$ or 14.4 cm long.

The length is a fraction! $\quad 14.4 = \dfrac{72}{5}$

Can every line segment be measured?

Irrational numbers

Already within the Pythagorean theorem lurks a profound mystery



By the Pythagorean theorem, c is a number that satisfies $c^2 = 1^2 + 1^2 = 2$

Is there such a number?

Assume $c = \frac{m}{n}$. Any fraction can be reduced to to simplest terms (e.g. $\frac{3}{6} = \frac{1}{2}$) Hence $\gcd(m,n) = 1$.

$$2 = c^2 = \left(\frac{m}{n}\right)^2 = \frac{m^2}{n^2} \implies 2n^2 = m^2$$

Therefore 2 divides $m^2 \implies 2 \mid m^2 \implies 2 \mid m$

$\implies m = 2k$ for some integer k.

$$2n^2 = m^2 = (2k)^2 = 4k^2$$

$$\implies n^2 = 2k^2 \implies 2 \mid n^2 \implies 2 \mid n.$$

But if 2 divides n and 2 divides m, then $g(m,n) \geq 2$ and this is a contradiction.

Thus $\sqrt{2}$ cannot be a fraction.

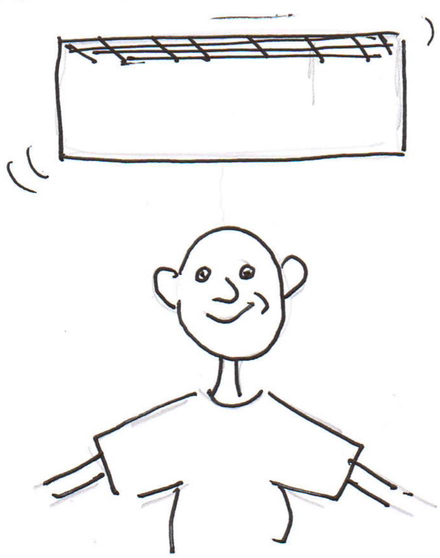
When you press $\sqrt{2}$ in your calculators you get $\sqrt{2} = 1.414213562373 1$. But this number cannot possibly be $\sqrt{2}$! No matter how many digits you write, the number you obtain is a fraction!

A similar calculation shows that $\sqrt{3}$, $\sqrt{5}$, $\sqrt{6}$, $\sqrt{7}$ etc. are not fractions.

<u>Q.</u> Why do I care?

<u>A.</u> The following reply was inspired by my dear students:

If you don't care about the nature of $\sqrt{5}$, you should be completely indifferent if I drop this heavy brick on your head.



The brick will travel a distance of $16t^2$ ft. in the t seconds that it will be in motion.

Am I liable for what will happen next?

Let's drop the brick 5 ft above your head. The brick will make contact with your skull at t satisfying $16t^2 = 5$ or $t = \frac{\sqrt{5}}{4}$

If $\sqrt{5}$ does not exist, the brick will never touch you! Hence, either nothing will happen to you or your skull will cave in, but not because of the brick! At any rate, I did all I could.

Calculating Roots

There are many techniques to find good approximations to $\sqrt{\cdot}$. One is particularly simple:

<u>Ex.</u> Find an approximation to $\sqrt{2}$.

<u>Solution:</u> $\sqrt{2} = 1 + \boxed{(\sqrt{2}-1)} = 1 + \dfrac{(\sqrt{2}-1)(\sqrt{2}+1)}{(\sqrt{2}+1)}$

$= 1 + \dfrac{1}{\sqrt{2}+1} = 1 + \dfrac{1}{2 + \boxed{(\sqrt{2}-1)}}$

Thus $\sqrt{2} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{\ddots}}}}$

we can use this continued fraction to approximate $\sqrt{2}$. The calculator estimates $\sqrt{2} = 1.414\,213\,562\,3731$

1) $1 + \boxed{\dfrac{1}{2}} = 1.5$

2) $1 + \dfrac{1}{2 + \boxed{\frac{1}{2}}} = 1 + \boxed{\dfrac{2}{3}} = \underbrace{1.}666\ldots$

3) $1 + \dfrac{1}{2 + \boxed{\frac{2}{3}}} = 1 + \boxed{\dfrac{3}{8}} = \underbrace{1.}375$

4) $1 + \dfrac{1}{2 + \boxed{\frac{3}{8}}} = 1 + \boxed{\dfrac{8}{19}} = \underbrace{1.4}21052\,631\,578\,95\ldots$

5) $1 + \dfrac{1}{2 + \boxed{\frac{8}{19}}} = 1 + \boxed{\dfrac{19}{46}} = \underbrace{1.41}3\,043\,478\,260\,87\ldots$

6) $1 + \dfrac{1}{2 + \boxed{\frac{19}{46}}} = 1 + \boxed{\dfrac{46}{111}} = \underbrace{1.414}\,414\,414\,414\,41\ldots$

7) $1 + \dfrac{1}{2 + \boxed{\frac{46}{111}}} = 1 + \boxed{\dfrac{111}{268}} = \underbrace{1.414}\,179\,109\,477\,61\ldots$

8) $1 + \dfrac{1}{2 + \boxed{\frac{111}{268}}} = 1 + \boxed{\dfrac{268}{647}} = \underbrace{1.414\,21}9\,474\,497\,68\ldots$

9) $1 + \dfrac{1}{2 + \boxed{\dfrac{268}{647}}} = 1 + \boxed{\dfrac{647}{1562}} = \underline{1.414212}\,548\ldots$

10) $1 + \dfrac{1}{2 + \boxed{\dfrac{647}{1562}}} = 1 + \dfrac{1562}{3771} = \underline{1.414213}\,736\,409\,44\ldots$

There is something poetic or enlightening about this derivation.

Deep meaning is tautology carried out to infinity.

Try to find approximations to

(a) $\sqrt{3}$ (b) $\sqrt{5}$ (c) $\sqrt{7}$