Derivatives Lecture 5

The Product and Quotient Rules

Ex. Find the derivative $\frac{d}{dx}(x^2 e^x)$

Solution: We know that $\frac{d}{dx}(x^2) = 2x$ and $\frac{d}{dx}(e^x) = e^x$. Hence $\frac{d}{dx}(x^2 e^x)$

$$= \frac{d}{dx}(x^2)\frac{d}{dx}(e^x) = 2x\,e^x.$$

Right?

Wrong!!! If you don't understand why an operation works and how it works, you will make an error! Not to mention that you are just filling your head with nonesense. Wasteful.

The geometric approach

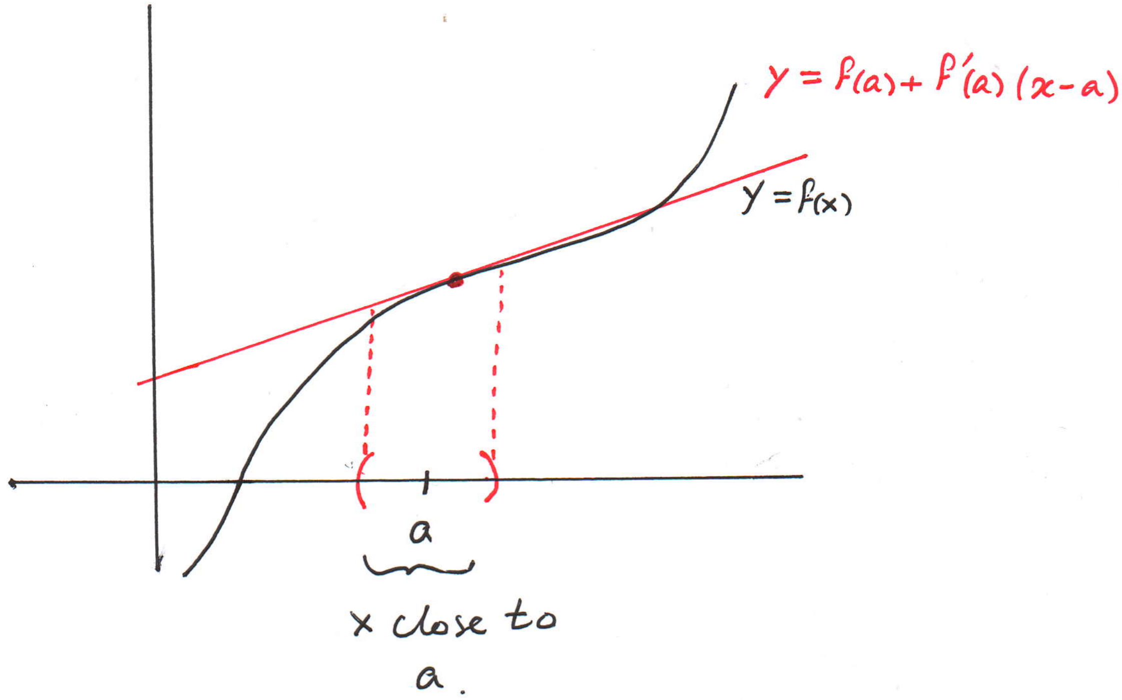
Zoom in on the graph of a differentiable function $y = f(x)$ at $(a, f(a))$. What will you see?

You will see the line $y - f(a) = f'(a)(x-a)$.

This line is a "stunt double". (or as I call it Каскадёр in Russian. This is not a Russian word in origin, but Kaska - helmet and der - puller makes Kaskader - helmet puller sound like what the word describes)

A stunt double for the more complicated "primadonna" function $f(x)$.

That is $f(x) \approx y(x) = f(a) + f'(a)(x-a)$



<u>Ex.</u> Approximate $\sqrt{9.01}$.

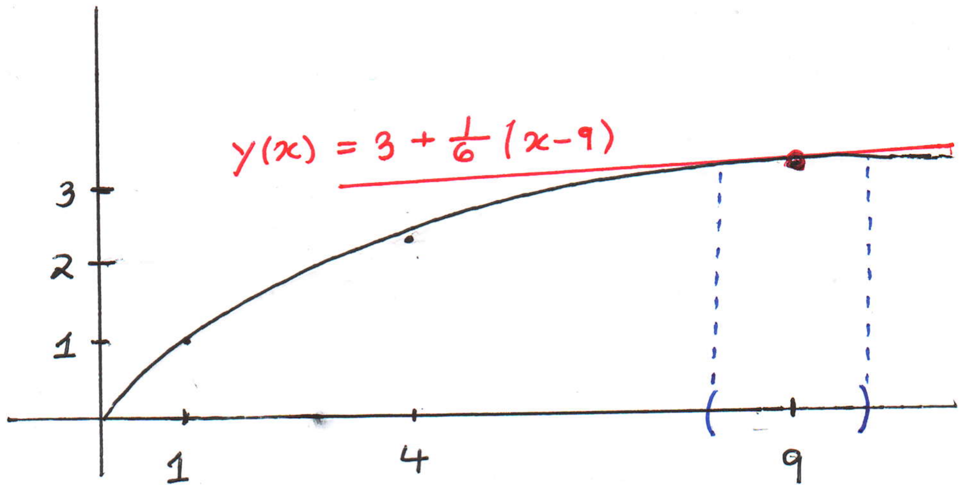
<u>Solution:</u> Let $f(x) = \sqrt{x}$. Then

$$f'(x) = \lim_{z \to x} \frac{\sqrt{z} - \sqrt{x}}{z - x} = \lim_{z \to x} \frac{\sqrt{z} - \sqrt{x}}{(\sqrt{z})^2 - (\sqrt{x})^2}$$

$$= \lim_{z \to x} \frac{\cancel{(\sqrt{z} - \sqrt{x})}}{\cancel{(\sqrt{z} - \sqrt{x})}(\sqrt{z} + \sqrt{x})}$$

$$= \lim_{z \to x} \frac{1}{\sqrt{z} + \sqrt{x}} = \frac{1}{2\sqrt{x}}$$

This means that as we zoom in at the point $(9, \sqrt{9})$ on the graph $y = \sqrt{x}$, we see the line $y(x) = \sqrt{9} + \frac{1}{2\sqrt{9}}(x-9)$



Do you see? 9.01 is close to 9. It is difficult to notice the difference between the red line $y(x) = 3 + \frac{1}{6}(x-9)$ and $f(x) = \sqrt{x}$ within the blue region. Thus $y(x) = 3 + \frac{1}{6}(x-9)$ can function as a stunt double for $\sqrt{x}$.

In particular $\sqrt{9.01} \approx 3 + \frac{1}{6}(9.01-9)$ $= 3 + \frac{1}{6}(0.01) = 3.001\overline{6}$. Contrast this with the calculator: $\sqrt{9.01} = 3.00166620396073\ldots$

We can use the linear approximation (stunt double) to figure out the derivative $\frac{d}{dx}(f(x)g(x))$.

<u>Thm:</u> Let $f(x)$ and $g(x)$ be differentiable at $x = a$. Then $\frac{d}{dx}\left[f(x)g(x)\right]\Big|_{x=a}$

$= f'(a)g(a) + f(a)g'(a)$.

<u>Proof:</u>

<u>1. Using geometry</u>

If we zoom in on the curve $y = f(x)g(x)$ at $(a, f(a)g(a))$ what line will we see?

$f(x) \approx f(a) + f'(a)(x-a) = f(a) + f'(a)\,dx$

$g(x) \approx g(a) + g'(a)(x-a) = g(a) + g'(a)\,dx$

Thus $f(x)g(x) \approx \left(f(a) + f'(a)\,dx\right)\left(g(a) + g'(a)\,dx\right)$

$= f(a)g(a) + \left[f'(a)g(a) + f(a)g'(a)\right]dx + f'(a)g'(a)(dx)^2$

when x is within 10^{-6} units away from a dx is a micron

How large will $(dx)^2$ be? If you look at dx through a microscope that magnifies the image a million times, dx will look 1 meter long. $(dx)^2 = (10^{-6})^2 = 10^{-12}$ will then look like it is the length of a microne! It will be invisible.

Hence the term $f'(a)g'(a)(dx)^2$ will become invisible.

The line that we will see is

$$\underbrace{f(a)g(a)}_{\text{y coordinate}} + \left[\underbrace{f'(a)g(a) + f(a)g'(a)}_{\text{slope.}}\right]dx$$

Thus the derivative = slope of tangent line

$= f'(a)g(a) + f(a)g'(a).$

2. By definition

$$\frac{d}{dx}\left(f(x)g(x)\right) = \lim_{z \to x} \frac{f(z)g(z) - f(x)g(x)}{z - x}$$

In that expression we are searching for

$$\lim_{z \to x} \frac{f(z)-f(x)}{z-x} = f'(x) \quad \text{and} \quad \lim_{z \to x} \frac{g(z)-g(x)}{z-x}$$

$= g'(x)$.

Now $$\lim_{z \to x} \frac{f(z)g(z) - f(x)g(x)}{z-x}$$

$$= \lim_{z \to x} \frac{f(z)g(z) - f(x)g(z) + f(x)g(z) - f(x)g(x)}{z-x}$$

$$= \lim_{z \to x} \left(g(z) \frac{f(z)-f(x)}{z-x} + f(x) \frac{g(z)-g(x)}{z-x} \right)$$

$$= \lim_{z \to x} g(z) \lim_{z \to x} \frac{f(z)-f(x)}{z-x} + \lim_{z \to x} f(x) \lim_{z \to x} \frac{g(z)-g(x)}{z-x}$$

$$= g(x)f'(x) + f(x)g'(x) = f'(x)g(x) + f(x)g'(x).$$

<u>Remark:</u> The product rule formula is easy to remember; the derivative travels from f to g

one step at a time.

<u>Ex.</u> Calculate $\frac{d}{dx}(x^2e^x)$

<u>Solution:</u>

$$\frac{d}{dx}\left(\overset{f}{x^2}\overset{g}{e^x}\right) = \overset{f'}{2x}\overset{g}{e^x} + \overset{f}{x^2}\overset{g'}{e^x}$$

<u>Ex.</u> (a) For any $n \in \mathbb{N}$ find $\frac{d}{dx}\left(x^{\frac{1}{n}}\right)$

(b) Find $\frac{d}{dx}\left(x^{1/3} \cdot x^2\right)$

(c) Find $\frac{d}{dx}\left(x^{1/3} \cdot x^2 + 2x^5 e^x\right)$

<u>Solution:</u>

(a) Let $f(x) = x^{1/n}$. Then

$$f'(x) = \lim_{z \to x} \frac{f(z) - f(x)}{z - x} = \lim_{z \to x} \frac{z^{1/n} - x^{1/n}}{z - x}$$

$$= \lim_{z \to x} \frac{z^{1/n} - x^{1/n}}{(z^{1/n})^n - (x^{1/n})^n} = \lim_{z \to x} \frac{\cancel{(z^{1/n} - x^{1/n})}}{\cancel{(z^{1/n} - x^{1/n})}\left((z^{1/n})^{n-1} + \dots + (x^{1/n})^{n-1}\right)}$$

$$= \frac{1}{(x^{1/n})^{n-1} + \dots + (x^{1/n})^{n-1}} = \frac{1}{n\, x^{\frac{n-1}{n}}}$$

$$= \frac{1}{n} \frac{1}{x^{1-\frac{1}{n}}} = \frac{1}{n} x^{\frac{1}{n}-1}.$$

Can you find this derivative geometrically?

(b) $\frac{d}{dx}\left(x^{1/3} \cdot x^2\right) = \frac{1}{3} x^{\frac{1}{3}-1} \cdot x^2 + x^{1/3} \cdot 2x.$

$$= \frac{1}{3} x^{\frac{1}{3}+1} + 2x^{\frac{1}{3}+1} = \left(\frac{1}{3}+2\right) x^{\frac{1}{3}+1}.$$

(c) $\frac{d}{dx}\left(x^{1/3} \cdot x^2 + 2x^5 e^x\right) = \left(\frac{1}{3}+2\right) x^{\frac{1}{3}+1}$

$+ 10x^4 e^x + 2x^5 e^x.$

Q. If $f(x), g(x), k(x)$ are differentiable, what do you think is the derivative of $f(x)g(x)k(x)$ in terms of the component functions and their derivatives?

A. $\frac{d}{dx}(f\cdot g\cdot k) = f'\cdot g\cdot k + f\cdot g'\cdot k + f\cdot g\cdot k'$

Do you see the pattern? The dash travels from one function to the next.

It is easy to prove that this pattern holds:

$(f\cdot g\cdot k)' = ([f\cdot g]\cdot k)' = [f\cdot g]'\cdot k + [f\cdot g]k'$

$= [f'\cdot g + f\cdot g']\cdot k + f\cdot g\cdot k' = f'\cdot g\cdot k + f\cdot g'\cdot k +$

$+ f\cdot g\cdot k'$.

Derivatives of a product of heigher order

Observe that

1) $(fg)' = f'g + fg'$

2) $(fg)'' = (f'g + fg')' = f''g + f'g' + f'g' + fg''$

$= f''g + 2f'g' + fg''$

3) $(fg)''' = (f''g + 2f'g' + fg'')' =$

$= f'''g + f''g' + 2f''g' + 2f'g'' + f'g'' + fg'''$

$= f'''g + 3f''g' + 3f'g'' + fg'''$

Looks familiar? Note that

1) $(x+y)^1 = x+y$

2) $(x+y)^2 = x^2+2xy+y^2$

3) $(x+y)^3 = x^3+3x^2y+3xy^2+y^3$

Higher order derivatives function like binomial expansion!

To show that symbolically, note that if ∂_f is the derivative of f and ∂_g is the derivative of g, then $(\partial_f+\partial_g)(f\cdot g) = \partial_f(f\cdot g)+\partial_g(f\cdot g)$

$= f'\cdot g + f\cdot g'$

Similarly $(\partial_f+\partial_g)^2 = \partial_f^2 + 2\partial_f\partial_g + \partial_g^2$.

Expansion for heigh powers $(\partial_f+\partial_g)^n$ can be accomplished quickly with the help of Pascal's Triangle:

$$
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & & 1 & & & (\quad)^1 \\
 & & 1 & & 2 & & 1 & & (\quad)^2 \\
 & 1 & & 3 & & 3 & & 1 & (\quad)^3 \\
 & & & & + & & & & \\
1 & & 4 & & 6 & & 4 & & 1 \quad (\quad)^4
\end{array}
$$

where the numbers in each raw are the sum of two numbers directly above from the previous raw (as shown by the red arrows).

Thus $(x+y)^4 = 1\cdot x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + 1\cdot y^4$.

<u>Ex.</u> Calculate $(x^5e^x)'''$

<u>Solution:</u> Let $f(x) = x^5$ and $g(x) = e^x$.

Denote by ∂_f and ∂_g the derivatives of f and g respectively.

Then $(\partial_f + \partial_g)^3 = \partial_f^3 + 3\partial_f^2\partial_g + 3\partial_f\partial_g^2 + \partial_g^3$

Now $\partial_f^3(x^5) = 5\cdot 4\cdot 3\, x^2 = 60x^2$

$\partial_f^2(x^5) = 5\cdot 4\, x^3 = 20x^3$

$\partial_f(x^5) = 5x^4$

and $\partial_g(e^x) = \partial_g^2(e^x) = \partial_g^3(e^x) = e^x$

Thus $(x^5e^x)''' = \partial_f^3(x^5)e^x + 3\partial_f^2(x^5)\partial_g(e^x)$
$+ 3\partial_f(x^5)\partial_g^2(e^x) + \partial_g^3(e^x) = 60x^2e^x + 60x^3e^x$
$+ 15x^4e^x + e^x.$

Quotient Rule

If $f(x)$ and $g(x)$ are differentiable, what is $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right)$?

As you should verify, $\left(\frac{f}{g}\right)' = \frac{gf' - g'f}{g^2}$

To remember the quotient rule, square the denominator, put the denominator to the top and multiply by the derivative of the numerator subtract, and have the place of the prime (')

reversed.

$$\frac{\boxed{gf'} - g'f}{\boxed{g^2}}$$

Ex. Compute $\frac{d}{dx}\left(\frac{x^5}{2e^x}\right)$

Solution:

$$\frac{\boxed{(2e^x)(x^5)'} - (2e^x)'x^5}{\boxed{(2e^x)^2}}$$

$$= \frac{2e^x \cdot 5x^4 - 2e^x \cdot x^5}{4e^{2x}}.$$

Ex. Compute $\frac{d}{dx}\left(\frac{1}{x^5}\right)$

Solution:

$$\frac{x^5(1)' - (x^5)' \cdot 1}{(x^5)^2} = \frac{-5x^4}{x^{10}}$$

$$= -5\frac{1}{x^6} = -5x^{-6}.$$

Ex. Compute $\frac{d}{dx} \dfrac{(2x^2+1)e^x + 10x}{5x^3 - 7x + 10}$

Solution: The main operation is quotient so we employ the quotient rule first. Notice however that other operations will be involved.

$$\frac{(5x^3-7x+10)\left[(2x^2+1)e^x+10x\right]' - (5x^3-7x+10)'\left[\,''\,\right]}{(5x^3-7x+10)^2}$$

$$= \frac{(5x^3-7x+10)\left((4x)e^x + (2x^2+1)e^x + 10\right) - (15x^2-7)\left[(2x^2+1)e^x+10x\right]}{(5x^3-7x+10)^2}$$