Derivatives Lecture 4

Are Exponential Funtions Polynomials?

In the previous section, we observed that if $f(x)=a^x$

then $f'(x) = \lim\limits_{h\to 0} \dfrac{a^{x+h}-a^x}{h} = \lim\limits_{h\to 0} \dfrac{a^x a^h - a^x}{h}$

$= a^x \lim\limits_{h\to 0} \dfrac{a^{0+h}-a^0}{h} = f(x)f'(0)$

where $f'(0) = \lim\limits_{h\to 0} \dfrac{a^h-1}{h}$ if the limit exists.

The limit $\lim\limits_{h\to 0} \dfrac{a^h-1}{h}$ is difficult to tackle directly.

Using $\dfrac{a^h-1}{h} \approx \lim\limits_{h\to 0} \dfrac{a^h-1}{h}$ when $|h|$ is small, we

can estimate

$$\lim_{h\to 0} \frac{2^h-1}{h} \approx \frac{2^{0.001}-1}{0.001} \approx 0.693$$

$$\lim_{h\to 0} \frac{3^h-1}{h} \approx \frac{3^{0.001}-1}{0.001} \approx 1.099$$

Since $f(x) = a^x$ is not a constant, $f(x)$ cannot have a horizontal line as its graph. Therefore $f'(x) = f'(0) f(x) \neq 0$. We cannot have $f'(0) = 0$.

<u>Q.</u> What is the simplest value we can hope for $f'(0) = \lim_{h \to 0} \frac{a^h - 1}{h}$?

<u>A.</u> If a is such that $\lim_{h \to 0} \frac{a^h - 1}{h} = 1$

$f'(x) = f'(0) f(x) = 1 \cdot f(x) = f(x)$. The simplest value for $f'(0)$ is 1.

<u>Hypothesis:</u> There exists some number $2 < e < 3$ such that $\lim_{h \to 0} \frac{e^h - 1}{h} = 1$.

<u>Remark:</u> In observing that $\lim_{h \to 0} \frac{2^h - 1}{h} \approx 0.693$ and $\lim_{h \to 0} \frac{3^h - 1}{h} \approx 1.099$, we

1) Assumed these limits converge. We don't know that!

2) Used estimates of $2^{0,001} = 2^{\frac{1}{1000}}$. Can you even estimate $\sqrt{2}$? How does your calculator do that?

In mathematics as perhaps in life, santa Clous exists as long as you believe in him.

To find e, we are led to consider the differential equation

$$\boxed{\begin{aligned} f'(x) &= f(x) \\ f(0) &= 1. \end{aligned}}$$

you may think of $f(x)$ as e^x (exponential function), but this is not necessary.

<u>Q.</u> Could any polynomial function satisfy the differential equation $f'(x) = f(x)$; $f(0) = 1$?

Is e^x (assuming the number e exists) a polynomial?

A: On first glance, e^x looks like it is a polynomial because it uses multiplication:

$$e^3 = e \cdot e \cdot e$$

However, $e^{\frac{1}{2}} = \sqrt{e}$ and we are unsure how to simulate square roots. More generally, we don't even know the meaning of $e^{\sqrt{2}}$. Could e^x be a polynomial?

Let $p(x) = a_0 + a_1 x + a_2 x^2$.

Then $p'(x) = a_1 + 2a_2 x$

$$p''(x) = 2a_2$$

$$p'''(x) = 0$$

But any $f(x)$ that satisfies $f'(x) = f(x)$, we also have

$$f''(x) = f(x)$$

$$f'''(x) = f(x)$$

Thus no quadratic polynomial can equal its derivative.

Hence $e^x \neq a_0 + a_1 x + a_2 x^2$.

Could e^x equal $p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$?

Well $p'(x) = a_1 + 2a_2 x + 3a_3 x^2$

$p''(x) = 2a_2 + 3\cdot 2\cdot a_3 \cdot x$

$p'''(x) = 3\cdot 2\cdot 1\cdot a_3$

$p^{(iv)}(x) = 0.$

But if $f'(x) = f(x)$, $f^{(iv)}(x) = f(x)$. Thus

$f(x)$ cannot be a cubic polynomial.

It is easy to see that if $p(x) = a_0 + a_1 x + \ldots + a_n x^n$,

then $p^{(n+1)}(x) = 0$. Hence functions like e^x

cannot be finite polynomials.

But what if we really really $\boxed{\text{want}}$ e^x to

be a polynomial? If you know how to ask

the math fairy, your wish may come true.

What if we try an infinite polynomial?

$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots + a_n x^n + \ldots$

Before we proceed with the argument, recall that for every positive integer n, $n! = n\cdot(n-1)(n-2)\cdot\ldots\cdot 1$.

For example, $3! = 3\cdot 2\cdot 1$ and $5! = 5\cdot 4\cdot 3\cdot 2\cdot 1$.

$0! \stackrel{\text{def}}{=} 1$.

If $f'(x) = f(x)$; $f(0) = 1$, how do we figure out the coefficients $a_0, a_1, a_2, \ldots, a_n$?

$$1 = f(0) = a_0 + \underbrace{a_1\cdot 0 + a_2\cdot 0^2 + a_3\cdot 0^3 + \ldots + a_n\cdot 0^n + \ldots +}_{=0}$$

Thus $\boxed{a_0 = 1}$

Notice that

$$f'(x) = \left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots + a_n x^n + \ldots +\right)'$$

$$= a_1 + 2a_2 x + 3a_3 x^2 + \ldots + n a_n x^{n-1} + \ldots +$$

$$= f(x)$$

isolates a_1.

$1 = f(0) = f'(0) = a_1$ or

$$\boxed{a_1 = 1}$$

Taking the derivative again

$$f''(x) = \left(a_1 + 2a_2 x + 3a_3 x^2 + \ldots + n a_n x^{n-1} + \ldots + \right)'$$

$$= 2a_2 + 3 \cdot 2 a_3 x + \ldots + n(n-1) a_n x^{n-2} + \ldots +$$

$$= f(x)$$

Thus $1 = f(0) = 2a_2$ and hence

$$\boxed{a_2 = \frac{1}{2}}$$

Similarly, $f'''(0) = 3 \cdot 2 \cdot 1\, a_3 = f(0) = 1$. Hence

$$\boxed{a_3 = \frac{1}{3!}}$$

In general, $f^{(n)}(x) = n!\, a_n + (n+1)(n) \cdots 2\, a_{n+1} x + \ldots +$

Hence $f^{(n)}(0) = n!\, a_n = f(0) = 1$. So

$$\boxed{a_n = \frac{1}{n!}}$$

This means that $f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} x^n =$

$$1 + x + \frac{1}{2} x^2 + \frac{1}{3!} x^3 + \frac{1}{4!} x^4 + \ldots +$$

should solve the equation $f'(x) = f(x)$; $f(0) = 1$.

Does it work?

$$f'(x) = \left(1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \ldots + \frac{1}{(n-1)!}x^{n-1} + \frac{1}{n!}x^n + \ldots +\right)'$$

$$= \left(1 + \frac{2}{2}x + \frac{3}{3!}x^2 + \frac{4}{4!}x^3 + \ldots + \frac{n-1}{(n-1)!}x^{n-2} + \frac{n}{n!}x^{n-1} + \ldots +\right)$$

$$= 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^2 + \ldots + \frac{1}{(n-2)!}x^{n-2} + \frac{1}{(n-1)!}x^{n-1} + \ldots +$$

$$= f(x).$$

We have a way to compute e now!

$$\boxed{e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \ldots + \frac{1}{n!}x^n + \ldots +}$$

so $e = 1 + 1 + \frac{1}{2} + \frac{1}{3!} + \ldots + \frac{1}{n!} + \ldots +$

or $e \approx 1 + 1 + \frac{1}{2} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} = 2 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120}$

≈ 2.71

Trying this with a calculator

$$\frac{(2.71)^{0.001} - 1}{0.001} \approx 0.997$$

If we add one more term and approximate e as

$$e \approx 1+1+\frac{1}{2}+\frac{1}{6}+\frac{1}{24}+\frac{1}{120}+\frac{1}{720} \approx 2.718$$

We get

$$\frac{(2.718)^{0.001}-1}{0.001} \approx 1.000396$$

The technique featured to solve the differential equation $f'(x) = f(x)$ is called Taylor series

In general, if $f(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n + \ldots +$

Then $a_n = \frac{f^n(0)}{n!}$ and $f(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 +$

$$+\ldots+ \frac{f^n(0)}{n!}x^n + \ldots + = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

Remark: These ideas are extreemly useful and powerful. Learn them as soon as possible!

Optional

The function $f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ is taken as the solution to the differential equation $f'(x) = f(x)$; $f(0) = 1$. How do we know that $f(x)$ defines an exponential function?

Before reading on, consult my lecture notes on combinatorial analysis (Stat 311). You need to study the combinatorial properties of multiplication and learn about the binomial theorem

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

$$= y^n + nxy^{n-1} + \frac{n(n-1)}{2} x^2 y^{n-2} + \ldots + x^n$$

where $$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

To prove that the exponent law $e^{x+y} = e^x e^y$ works for all real values x and y, observe that

$$e^x e^y = \left(1 + x + \frac{x^2}{2} + \ldots + \right)\left(1 + y + \frac{y^2}{2} + \ldots + \right)$$

can be grouped by adding powers of the form $x^k y^{n-k}$ such that the sum of the x-power and the y-power add up to n:

$$e^x e^y = \underbrace{\boxed{1 \cdot 1}}_{n=0} + \underbrace{\boxed{1 \cdot y + xy}}_{n=1} + \underbrace{\boxed{\frac{1 \cdot y^2}{2} + xy + \frac{x^2 \cdot 1}{2}}}_{n=2} + \ldots +$$

In particular if $f(x) = e^x$, then

$$f(x)f(y) = \sum_{n=0}^{\infty}\left(\frac{1\cdot y^n}{n!} + \frac{x\,y^{n-1}}{1!\,(n-1)!} + \ldots + \frac{x^{n-1}y}{(n-1)!\,1!} + \frac{x^n\cdot 1}{n!}\right)$$

$$= \sum_{n=0}^{\infty}\frac{1}{n!}\left(y^n + \frac{n!}{1!(n-1)!}\,x\,y^{n-1} + \ldots + \frac{n!}{(n-1)!\,1!}\,x^{n-1}y + x^n\right)$$

$$= \sum_{n=0}^{\infty}\frac{1}{n!}(x+y)^n = f(x+y).$$

This allows us to observe that

$$f(p) = f(\underbrace{1+1+\ldots+1}_{p \text{ times}}) = f(1)f(1)\cdot\ldots\cdot f(1) = f(1)^p$$

for any $p \in \mathbb{N}$

Also $\quad 1 = f(0) = f(p+[-p]) = f(p)\cdot f(-p)$ or

$$f(-p) = \frac{1}{f(p)} = \frac{1}{f(1)^p}$$

Likewise $\quad f(\tfrac{1}{q})^q = f(\underbrace{\tfrac{1}{q}+\tfrac{1}{q}+\ldots+\tfrac{1}{q}}_{q \text{ times}}) = f(1)$

Thus upon taking q^{th} roots

$$f(\tfrac{1}{q}) = f(1)^{\frac{1}{q}}.$$

In general $\quad f(r) = f(1)^r \quad$ for any $r \in \mathbb{Q}$.

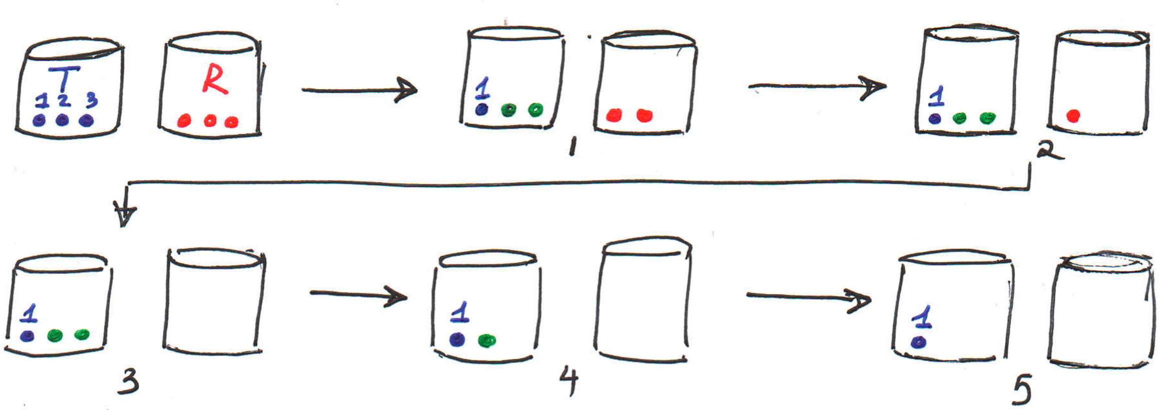
This establishes $f(x)$ as an exponential function.

Important Exponential limits.

Playing with the derivative reveals surprising connections to the world at large.

Ex. A good Russian man never goes a day without tea or alcohol. Consider the following drinking habit: Begin with a glass of tea and idertical glass of rum. Take a sip from the cup of tea, pour in the rum to make the glass full again, mix the spiked tea thoroughly, and take a sip again. Continue doing so until both the rum and are gone. What is the probability that the last sip is tea?

Solution: Imagine that we sample 1 molecule at a time. To understand the process, suppose we initially had only 3 molecules of tea and only 3 molecules of rum.



Let E be the event that the last molecule sampled is tea. Clearly $p(E) = p(E_1) + p(E_2) + p(E_3) = 3P(E_1)$, where E_k = event that the last sampled molecule is k $(k = 1, 2, 3)$.

Focus on molecule #1. It is the last molecule sampled if it survives 5 samplings.

$$p(E_1) = \frac{2}{3} \cdot \frac{2}{3} \cdot \frac{2}{3} \cdot \frac{2}{3} \cdot \frac{1}{2} = \left(\frac{2}{3}\right)^3 \cdot \frac{1}{3}$$

Hence $P(E) = 3P(E_1) = \left(\frac{2}{3}\right)^3 = \left(1 - \frac{1}{3}\right)^3$

you can easily verify that if you begin with 4 molecules in each glass, $P(E) = \left(\frac{3}{4}\right)^4 = \left(1 - \frac{1}{4}\right)^4$.

In general there are many billions of molecules in each beaker so $\lim_{n \to \infty} \left(1 - \frac{1}{n}\right)^n$ is a good estimate of what will happen then.

Q. What do you think is the value of

$$\lim_{n \to \infty} \left(1 - \frac{1}{n}\right)^n ?$$

A. This is not at all a simple limit! The answer happens to be e^{-1}! Here is how this may be seen to come about.

If $f'(x) = f(x)$ and $f(0) = 1$, then appealing to the definition of the derivative

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = f(x)$$

Thus if h is small

$$\frac{f(x+h) - f(x)}{h} \approx f(x)$$

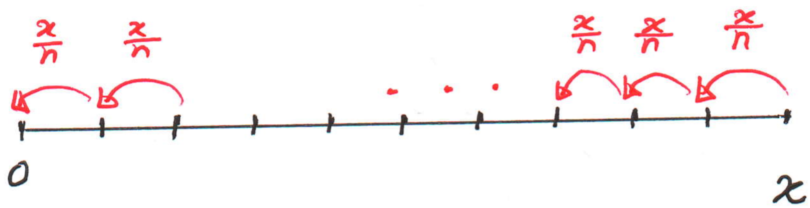
Hence $\quad f(x+h) - f(x) \approx h f(x)$

$$f(x+h) \approx f(x) + h f(x) = (1+h) f(x)$$

This means $f(\#) \approx (1 + \text{small step}) f(\# - \text{small step})$

In symbols $f(x) \approx (1+h) f(x-h)$

We now trace our steps back to 0:



Divide the number line $[0, x]$ into n small subsegments, where n is very large.

The length of each segment is $\frac{x}{n}$.

Thus $f(x) \approx \left(1+\frac{x}{n}\right) f\left(x-\frac{x}{n}\right) \approx \left(1+\frac{x}{n}\right)\left(1+\frac{x}{n}\right) f\left(x-2\frac{x}{n}\right)$

Continuing in this fashion, we obtain

$$f(x) \approx \left(1+\frac{x}{n}\right)^n f\left(x-n\frac{x}{n}\right)$$

$$f(x) \approx \left(1+\frac{x}{n}\right)^n f(0)$$

$$f(x) \approx \left(1+\frac{x}{n}\right)^n$$

Thus we see that

$$f(x) = \lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^n$$

In particular, $\displaystyle\lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^n = e^x$.

<u>Ex.</u> Compute

(a) $\displaystyle\lim_{n \to \infty} \left(1 + \frac{5}{n}\right)^n$

(b) $\displaystyle\lim_{n \to \infty} \left(1 - \frac{3}{n}\right)^n$

(c) $\displaystyle\lim_{n \to \infty} \left(1 - \frac{1}{n}\right)^n$

(d) $\displaystyle\lim_{x \to \infty} \left(1 + \frac{\sqrt{2}}{n}\right)^n$

<u>Solution:</u>

(a) $\displaystyle\lim_{n \to \infty} \left(1 + \frac{5}{n}\right)^n = e^5$

(b) $\displaystyle\lim_{n \to \infty} \left(1 - \frac{3}{n}\right)^n = e^{-3}$

(c) $\displaystyle\lim_{n \to \infty} \left(1 - \frac{1}{n}\right)^n = e^{-1}$

(d) $\displaystyle\lim_{n \to \infty} \left(1 + \frac{\sqrt{2}}{n}\right)^n = e^{\sqrt{2}}$

The exponential function makes constant and not coincidental presense in all aspects of natural phenomena. If you are happy or sad, fearful, angry or indifferent, there is e^x lurking in the background.

Ex. If there are 40 people in the room, what is the probability that at least two of them share a birthday?

Solution: The probability that no two people share a birthday is

$$\frac{\binom{365}{40} 40!}{365^{40}} = \frac{365 \cdot 364 \cdot \ldots \cdot (365-40+1)}{365^{40}}$$

We can approximate this probability by conducting $\binom{40}{2}$ experiments that consist in compairing any two people and seeing if their birthdays match.

Probability of no matches

$$\underbrace{\left(1-\frac{1}{365}\right)}_{1}\underbrace{\left(1-\frac{1}{365}\right)}_{2}\cdots\underbrace{\left(1-\frac{1}{365}\right)}_{\frac{40\cdot 39}{2}}$$

Hence $P(\text{No match}) \approx \left(1-\frac{1}{365}\right)^{\frac{40\cdot 39}{2}}$

Notice that $\left(1-\frac{1}{365}\right)^{780} = \left[\left(1-\frac{1}{365}\right)^{365}\right]^{\frac{780}{365}}$

$\approx \left[e^{-1}\right]^{2.137} \approx 0.118$

Hence the probability of match is $1-0.118 = 0.882$ or 88 %.

Check the matching problem! Later in the probability course, you will see why e^x plays such a prominent role.