Derivatives Lecture 3

Derivatives of Polynomials and Exponential Functions

Polynomials

Recall that a polynomial is a function that only uses addition and multiplication operations.
(5−2 can be viewed as five plus −2).

For example $f(x) = 1-(x+2)^2 + ([x+2]-1)^3 + 5x - x^2$ only uses addition and multiplication.

We can rearange this expression and group it by the number of multiplications of the input $x^0, x^1, x^2,$ etc.

$$f(x) = 1-(x^2+4x+4) + (x^3+3x^2+3x+1) + 5x - x^2$$

$$= (1-4+1) + (-4+3+5)x + (-1+3-1)x^2 + x^3$$

$$= -2 + 4x + x^2 + x^3$$

You should convince yourself that every polynomial is of the form

$$p(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n$$

<u>Ex.</u> What is the derivative of a constant function $f(x) = c$?

<u>Solution:</u> A constant function is a horizontal line with slope 0. Hence $f'(x) = 0$

$$\boxed{\frac{d}{dx}(c) = 0}$$

<u>Ex.</u> Let $f(x) = x$. Find $f'(x)$.

<u>Solution:</u> $f'(x) = \lim\limits_{z \to x} \dfrac{f(z) - f(x)}{z - x}$

$$= \lim_{z \to x} \frac{z - x}{z - x} = 1.$$

Remark: $f(x) = x$ is a line of slope 1. No calculation was necessery.

Ex. Let $f(x) = x^2$. Find $f'(x)$

Solution: $f'(x) = \lim_{z \to x} \frac{z^2 - x^2}{z - x}$

$$= \lim_{z \to x} \frac{\cancel{(z-x)}(z+x)}{\cancel{(z-x)}} = \lim_{z \to x} (z + x) = 2x.$$

Ex. Let $f(x) = x^3$. Find $f'(x)$.

Solution: $\lim_{z \to x} \frac{z^3 - x^3}{z - x} = ???$

How do we factor $z^3 - x^3$

We can use long division

$$
\begin{array}{r|l}
 & z^2 + xz + x^2 \\
\hline
z - x & z^3 - x^3 \\
 & -\underline{\left(z^3 - xz^2\right)} \\
 & xz^2 - x^3 \\
 & -\underline{\left(xz^2 - x^2 z\right)} \\
 & x^2 z - x^3 \\
 & -\underline{\left(x^2 z - x^3\right)} \\
 & 0
\end{array}
$$

Hence $\displaystyle\lim_{z \to x} \frac{z^3 - x^3}{z - x} = \lim_{z \to x} \frac{\cancel{(z-x)}(z^2 + xz + x^2)}{\cancel{(z-x)}}$

$$= \lim_{z \to x} \left(z^2 + xz + x^2\right) = x^2 + x^2 + x^2 = 3x^2$$

Ex. Find the derivative of $f(x) = x^4$.

Solution: Clearly we must learn to factor $z^4 - x^4$. We can try long division again.

$$
\begin{array}{r|l}
 & z^3 + xz^2 + x^2z + x^3 \\ \hline
z - x & z^4 - x^4 \\
 & -\,(z^4 - xz^3) \\ \hline
 & xz^3 - x^4 \\
 & -\,(xz^3 - x^2z^2) \\ \hline
 & x^2z^2 - x^4 \\
 & -\,(x^2z^2 - x^3z) \\ \hline
 & x^3z - x^4 \\
 & -\,(x^3z - x^4) \\ \hline
 & 0
\end{array}
$$

Hence $f'(x) = \lim\limits_{z \to x} \dfrac{z^4 - x^4}{z - x}$

$$= \lim_{z \to x} \frac{\cancel{(z-x)}(z^3 + xz^2 + x^2z + x^3)}{\cancel{(z-x)}}$$

$$= x^3 + x^3 + x^3 + x^3 = 4x^3$$

Do you begin to notice the pattern?

How would you factor $z^n - x^n$?

$$z^n - x^n = (z-x)(z^{n-1} + x z^{n-2} + x^2 z^{n-3} + \ldots + x^{n-2} z + x^{n-1})$$

This pattern is easy to remember if you understood Oscar Wilde's short story "The Nightingale and the Rose":

z – the nightingale gives its life force to the rose x digit after digit until the nightingale is gone and the rose is red.

$$z^{n-1} + x z^{n-2} + x^2 z^{n-3} + \ldots + x^{n-1}$$

is a summary of this story.

For instance

$$z^6 - x^6 = (z-x)(z^5 + x z^4 + x^2 z^3 + x^3 z^2 + x^4 z + x^5)$$

Once we see the pattern, it is easy to verify that it is always true:

$$S = z^{n-1} + x z^{n-2} + \ldots + x^{n-1}$$

$$zS = z^n + x z^{n-1} + \ldots + x^{n-1} z$$

$$xS = x z^{n-1} + x^2 z^{n-2} + \ldots + x^n$$

Thus $zS - xS = (z-x)S = z^n - x^n$.

<u>Thm:</u> $\frac{d}{dx}(x^n) = nx^{n-1}$ if n is a positive integer.

<u>Proof:</u> Let $f(x) = x^n$, then $f'(x) = \lim_{z \to x} \frac{f(z)-f(x)}{z-x}$

$$= \lim_{z \to x} \frac{z^n - x^n}{z-x} = \lim_{z \to x} \frac{\cancel{(z-x)}(z^{n-1} + xz^{n-2} + \dots + x^{n-1})}{\cancel{(z-x)}}$$

$$= \lim_{z \to x} (\underbrace{\overset{0}{\downarrow} z^{n-1} + \overset{1}{\downarrow} x z^{n-2} + \overset{2}{\downarrow} x^2 z^{n-3} + \dots + \overset{n-1}{\downarrow} x^{n-1}}_{\text{Total } n \text{ summands}}) = nx^{n-1}$$

Each approaches x^{n-1}

<u>Ex.</u> $\frac{d}{dx}(x^6) = 6x^{6-1} = 6x^5$

$\frac{d}{dx}(x^{101}) = 101x^{100}$

$\frac{d}{dx}(x^{30}) = 30x^{29}$

We now have our first shortcut!

To quickly compute derivatives of polynomials, we will need the following results:

<u>Thm:</u> Let $f(x)$ and $g(x)$ be differentiable and let c be a constant. Then

1) $\frac{d}{dx}(cf(x)) = c\frac{d}{dx}(f(x)) = cf'(x)$

2) $\frac{d}{dx}\left(f(x)+g(x)\right) = \frac{d}{dx}(f(x)) + \frac{d}{dx}(g(x))$

$= f'(x) + g'(x)$

3) $\frac{d}{dx}\left(f(x)-g(x)\right) = \frac{d}{dx}(f(x)) - \frac{d}{dx}(g(x))$

$= f'(x) - g'(x)$

<u>Proof:</u>

1) $$\frac{d}{dx}(cf(x)) = \lim_{z\to x} \frac{cf(z)-cf(x)}{z-x}$$

$$= \lim_{z\to x} c\,\frac{f(z)-f(x)}{z-x} = cf'(x)$$

2) $$\frac{d}{dx}\left(f(x)+g(x)\right) = \lim_{z\to x} \frac{\left[f(z)+g(z)\right]-\left[f(x)+g(x)\right]}{z-x}$$

$$= \lim_{z \to x} \frac{\left(f(z) - f(x)\right) + \left(g(z) - g(x)\right)}{z - x}$$

$$= \lim_{z \to x} \left(\frac{f(z) - f(x)}{z - x} + \frac{g(z) - g(x)}{z - x} \right)$$

$$= \lim_{z \to x} \frac{f(z) - f(x)}{z - x} + \lim_{z \to x} \frac{g(z) - g(x)}{z - x} = f'(x) + g'(x)$$

3) This is established in the same way as 2.

<u>Ex.</u> By property 1, $\frac{d}{dx}\left(5x^3\right) = 5\frac{d}{dx}(x^3)$

$= 5 \cdot 3x^2 = 15x^2$

Combining Properties 1–3, we see that

$$\frac{d}{dx}\left(3 - 5x + 8x^3 + 10x^4\right)$$

$$= \frac{d}{dx}(3) - 5\frac{d}{dx}(x) + 8\frac{d}{dx}(x^3) + 10\frac{d}{dx}(x^4)$$

$$= 0 - 5 \cdot 1 + 8\left(3x^2\right) + 10\left(4x^3\right)$$

$$= -5 + 24x^2 + 40x^3.$$

Ex. Calculate $\frac{d}{dx}\left(10 - x^2 + 7x^3\right)$

Solution:

$$\frac{d}{dx}\left(10 - x^2 + 7x^3\right) = -2x + 27x^2.$$

Derivatives of exponential functions

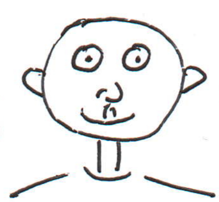
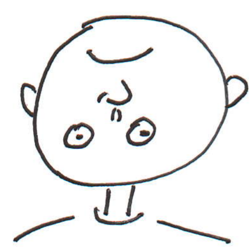
Ex. Calculate $\frac{d}{dx}\left(2^x\right)$.

Solution:

Many students write $\frac{d}{dx}\left(2^x\right) = x2^{x-1}$

This is Wrrrong!!!

You are confusing x^2 and 2^x.

That's like confusing  with 

and it makes me confuse you with

First, recall that $a^3 = a \cdot a \cdot a$ and in general

$$a^n = \underbrace{a \cdot a \cdot a \cdot \ldots \cdot a}_{n \text{ times.}}$$

Notice that $a^3 \cdot a^2 = (a \cdot a \cdot a) \cdot (a \cdot a) =$

$= a \cdot a \cdot a \cdot a \cdot a = a^{3+2}$

In general, $a^n \cdot a^m = a^{n+m}$ With some effort, this property of exponents can be extended to arbitrary real numbers x and y:

$a^x a^y = a^{x+y}$.

We can now attempt to differentiate

$f(x) = a^x$:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{a^{x+h} - a^x}{h}$$

$$= \lim_{h \to 0} \frac{a^x a^h - a^x}{h} = a^x \lim_{h \to 0} \frac{a^h - 1}{h}$$

$$= a^x \lim_{h \to 0} \frac{a^{0+h} - a^0}{h}$$

what does
this look like?

This is no longer a simple limit that we can do away with a simple algebraic trick. Instead, notice that we begun with the derivative at an arbitrary point x and ended up with the definition of the derivative at 0:

$$\lim_{h \to 0} \frac{a^{x+h} - a^x}{h} \implies \lim_{h \to 0} \frac{a^{0+h} - a^0}{h}$$

Thus, $\frac{d}{dx}(a^x)$ exists if and only if the function is differentiable at $x=0$, in which case,

$$f'(x) = a^x \lim_{h \to 0} \frac{a^{0+h} - a^0}{h} = a^x f'(0)$$

We haven't quite finished the calculation, but we see that the derivative isn't much changed.

$$f(x) = a^x \xrightarrow{\prime} \boxed{f'(0)} \cdot a^x \xrightarrow{\prime} [f'(0)]^2 a^x$$

number

$$\xrightarrow{\prime} [f'(0)]^3 a^x \quad \text{etc.}$$

In Newton's notation, we write f', f'', f''', f^{iv} and in general $f^{(n)}$ to indicate that n derivative operations have been carried out.

Thus, if $f(x) = a^x$

then $f'''(x) = [f'(0)]^3 a^x$ as above.

We can use the calculator to estimate the number $f'(0)$ for particular values of a:

<u>Ex.</u> If $a = 2$, $\displaystyle\lim_{h \to 0} \frac{a^{0+h} - a^0}{h}$

$$= \lim_{h \to 0} \frac{2^h - 1}{h} \approx \frac{2^h - 1}{h} \quad \text{when } h \text{ is small.}$$

Thus $\lim_{h \to 0} \frac{2^h - 1}{h} \approx \frac{2^{0.001} - 1}{0.001} \approx 0.69$

Hence

$$\frac{d}{dx}\left(2^x\right) \approx (0.69) \cdot 2^x$$

In general, $\left(2^x\right)^{(n)} \approx (0.69)^n \cdot 2^x$.

If $a = 3$, $\lim_{h \to 0} \frac{3^h - 1}{h} \approx \frac{3^{0.001} - 1}{0.001} \approx 1.099$

Hence

$$\frac{d}{dx}\left(3^x\right) \approx (1.099) \cdot 3^x$$

<u>Comprehension Check</u>

Estimate $\frac{d}{dx}\left([2.8]^x\right)$.

<u>Solution:</u> We need $\lim_{h \to 0} \frac{2.8^h - 1}{h} \approx \frac{2.8^{0.001} - 1}{0.001}$

≈ 1.03

Thus $\frac{d}{dx}\left([2.8]^x\right) \approx (1.03)\,[2.8]^x$.

<u>Remark</u>: We have cheated here a little (or in fact a lot). How does the calculator compute, say $\frac{2^{0.001}-1}{0.001}$? $2^{0.001} = 2^{\frac{1}{1000}}$

We bearly know how to calculate square roots. How are we to deal with 1000^{th} roots?

<u>Q.</u> We have seen that $\frac{d}{dx}(2^x) \approx \underbrace{(0.69)}_{C_2} 2^x$ and $\frac{d}{dx}(3^x) \approx \underbrace{(1.099)}_{C_3} 3^x$.

what would you have liked a to be to make $\frac{d}{dx}(a^x) = C_a a^x$ as simple as possible?

<u>A.</u> By playing with the value of a, it seems there might be a number $a = e$ such that $\frac{d}{dx}(e^x) = 1 \cdot e^x = e^x$.

Moreover, it appears that $2 < e < 3$

and, of course, e satisfies

$$\lim_{h \to 0} \frac{e^h - 1}{h} = 1.$$

The function $f(x) = e^x$ is one of the most significant functions in mathematics. In the next section, we shall explore it in great detail!