Derivatives Lecture 1

The Idea of the Derivative

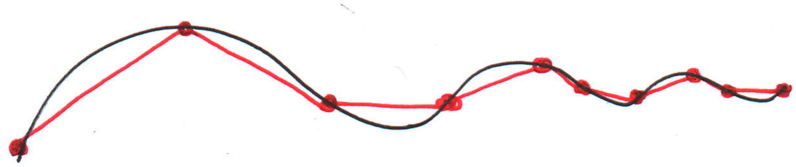
We now come to the first central problem of calculus

The Tangent Problem: Given a curve C how do we find the tangent line to the curve at some point (a,b) on the curve?

Q. Why is this a major problem? Why would anyone care?

A. 1) The simplest possible curve is a line.

A complicated curve could perhaps be analysed and described in terms of lines.

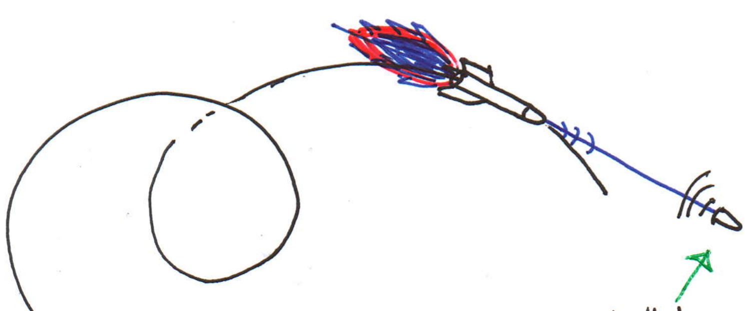


This analysis reveals many deep and practical insights you can see some of them in my Calculus III notes on dot product and work.

In general Calculus is the philosophy that regards complex phenomena as a kind of lego construction built out of simple blocks.

- An image on the TV is built out of many simple pixels
- Sound Wave constructed out of superposition of simple sine waves.
- Areas and volumes described as the sum of simple areas or simple volumes – those of rectangles and boxes.

2) According to Newton's first law, an object moves in a straight line with constant velocity when the net force on the object is 0.



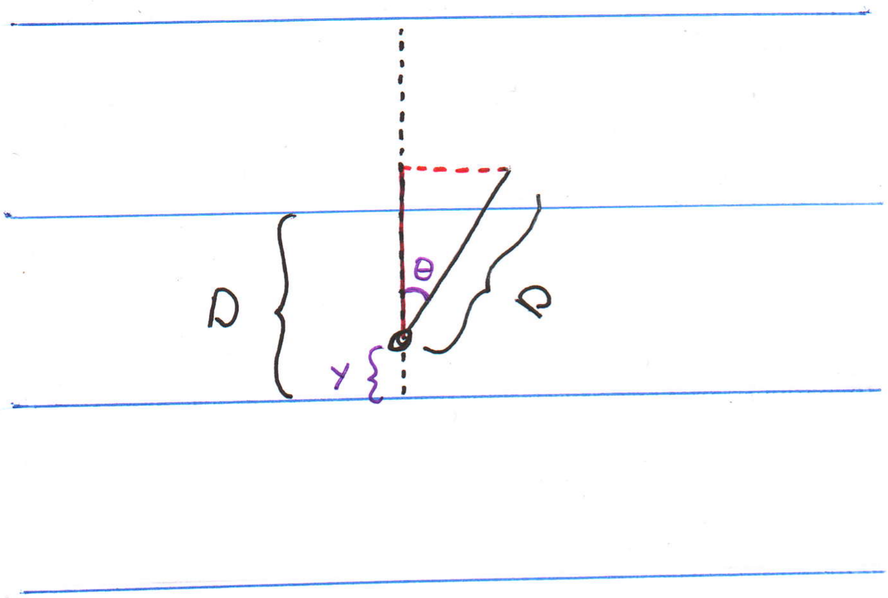
satellite will move along a tangent line to the tragectory of the rocket.

3) As we shall see, the tangent line problem is mathematically identical to the instantenious velocity problem with which our investigation of limits was started.

It is marvelous when you ask "What is this for?" "What are the statements about?" and "why would anyone be interested?" Just be aware that interesting ideas do not announce themselves directly. The most amazing jaw dropping results are never direct consequences that one sees at a glance of an idea that was just defined.

Ex. Here is an idea to compute π. Take a needle of length D and drop it randomly over a ruled sheet of paper in which the distance between the lines is D. What is the probability that the needle intersects one of the line?

Solution:



Needle intersects a horizontal line $\iff Y + D\cos\theta \geq D$

The desired probability is given by

$$P = \int_0^{\frac{\pi}{2}} \int_{D - D\cos\theta}^{D} \frac{2}{\pi D} \, dy \, d\theta = \int_0^{\frac{\pi}{2}} \frac{2}{\pi} \cos\theta \, d\theta = \frac{2}{\pi}$$

Hence by dropping the needle multiple times and counting

$$\frac{\#\text{ times needle crosses line}}{\#\text{ total needles thrown.}} \longmapsto \frac{2}{\pi}$$

Naturally, I'm not expecting you to understand the solution. Note only the number of concepts involved:

- Convergence of infinite sequences $\lim\limits_{n \to \infty} \frac{n(N)}{n}$.

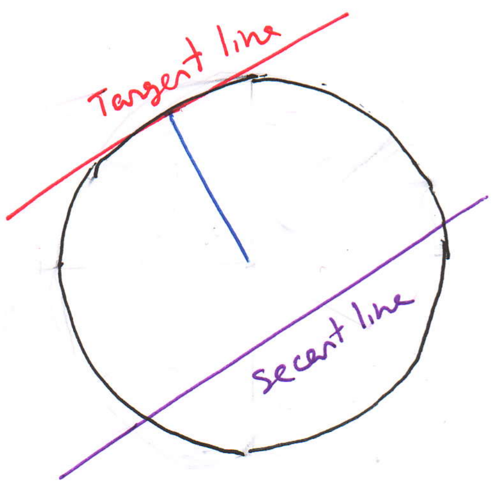
- Trigonometry

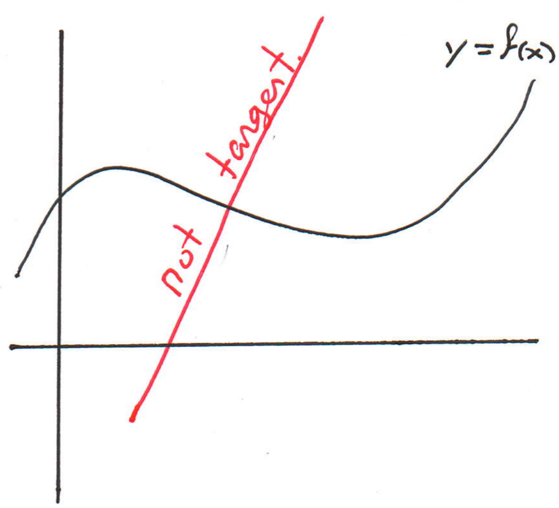
- Anti-derivatives (and for that we must know derivatives!!!)

- Much, much more.

Back to the tangent problem!

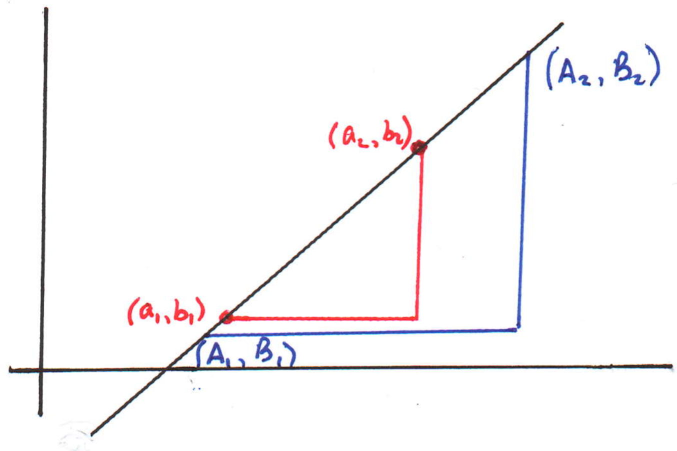
Given a curve $y = f(x)$, how do we construct tangent lines? For circles, we can define a tangent line as any line, which intersects the circle exactly in one point. This definition isn't good for other curves.





instead we can think as follows.

First, recall the idea of slope. Once we fix a perspective, we can describe the steepness of a line as a ratio of rise over run



Notice that the red and blue triangles are similar. Hence the ratio of vertical to horizontal length is the same for both triangles.

$$m = \frac{b_2 - b_1}{a_2 - a_1} = \frac{B_2 - B_1}{A_2 - A_1}$$

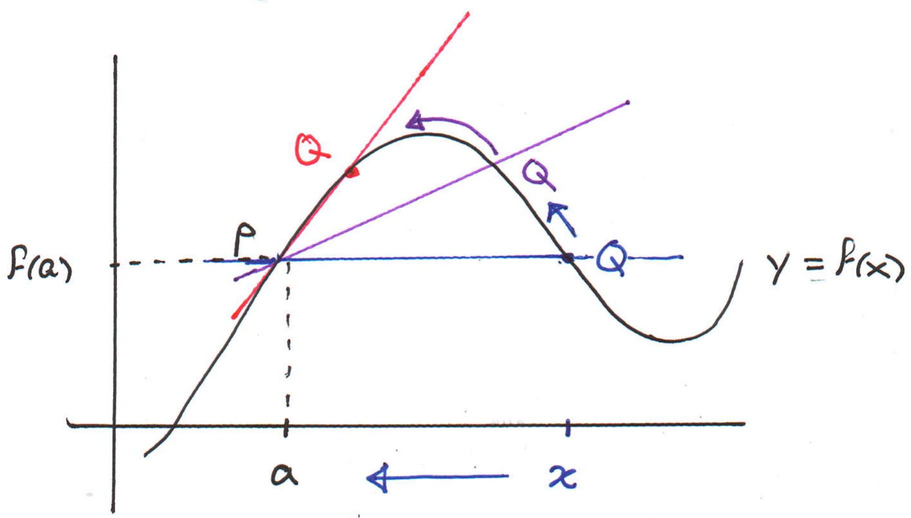
If the slope m and a point (a, b) on the line are known, another point (x, y) is on the line if and only if

$$\frac{y - b}{x - a} = m.$$

The equation of this line can be written as

$$y - b = m(x - a).$$

Through the curve $y = f(x)$, draw a secant line that goes through the points $(a, f(a))$ and $(x, f(x))$



The secant line joining P and Q has slope

$$m_{sec} = \frac{f(x) - f(a)}{x - a}.$$

by moving x closer to a, the secant line cuts less and less into the curve $y = f(x)$ and the line starts to look like the tangent.

In particular

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = m_{tan}.$$

Hint: Think of the secant line as the trajectory of a bullet that enters the curve at P and exit at Q. If Q is very close to P, the bullet essentially grazes the curve.

Ex. Find the equation of the tangent line to the curve $y = x^2$ at $x = 2$

Solution:

We want the tangent line to go through the point $(2, 2^2) = (2, 4)$. We need to know the slope.

$$m = \lim_{x \to 2} \frac{x^2 - 4}{x - 2} = \lim_{x \to 2} \frac{\cancel{(x-2)}(x+2)}{\cancel{(x-2)}}$$

$$= \lim_{x \to 2} (x+2) = 4$$

Hence an equation of the tangent line (the same line has infinitely many equations that define it) is

$$\boxed{y - 4 = 4(x-2).}$$

<u>Ex.</u> Find the tangent line to the curve $y = f(x)$, where $f(x) = -3x^2 + x + 5$ at $x = 1$. Can you write the general equation for the point $(a, f(a))$?

<u>Solution:</u>

We want to find an equation of the tangent line at the point $(1, f(1)) = (1, 3)$

$$m = \lim_{x \to 1} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1} \frac{-3x^2 + x + 5 - 3}{x - 1}$$

$$= \lim_{x \to 1} \frac{-3x^2 + x + 2}{x - 1} = \lim_{x \to 1} \frac{(-3x-2)\cancel{(x-1)}}{\cancel{(x-1)}}$$

$$= \lim_{x \to 1} (-3x-2) = -3-2 = -5$$

Thus an equation is

$$\boxed{y - 3 = -5(x-1).}$$

The general formula is

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{x \to a} \frac{(-3x^2+x+5)-(-3a^2+a+5)}{(x-a)}$$

$$= \lim_{x \to a} \frac{-3(x^2-a^2) + (x-a)}{(x-a)} = \lim_{x \to a} \frac{-3(x+a)(x-a) + (x-a)}{(x-a)}$$

$$= \lim_{x \to a} \frac{[-3(x+a)+1]\cancel{(x-a)}}{\cancel{(x-a)}} = -3(2a)+1$$

$$= -6a+1. \qquad \boxed{y - (-3a^2+a+5) = (-6a+1)(x-a)}$$

Ex. Find the tangent line to the curve $y = \sqrt{x}$ at $x = 9$.

Solution:

The slope is given by

$$\lim_{x \to 9} \frac{\sqrt{x} - \sqrt{9}}{x - 9} = \lim_{x \to 9} \frac{(\sqrt{x} - 3)(\sqrt{x} + 3)}{(x - 9)(\sqrt{x} + 3)}$$

$$= \lim_{x \to 9} \frac{\cancel{(x - 9)}}{\cancel{(x - 9)}(\sqrt{x} + 3)} = \frac{1}{\sqrt{9} + 3} = \frac{1}{6}$$

Hence an equation of the tangent line

$$\boxed{y - 3 = \frac{1}{6}(x - 9)}$$

Ex. A particle moves so that its position at time t is $x(t) = 2t^2 - t + 1$

Find the velocity of the particle at time $t = 1$.

Solution:

Recall that if t is close to 1, then there would not have been much time for the velocity to change due to acceleration. Hence

$$v(1) = \lim_{t \to 1} \frac{x(t) - x(1)}{t-1} = \lim_{t \to 1} \frac{(2t^2 - t + 1) - (2)}{t-1}$$

$$= \lim_{t \to 1} \frac{2t^2 - t - 1}{t-1} = \lim_{t \to 1} \frac{(2t+1)\cancel{(t-1)}}{\cancel{(t-1)}}$$

$$= \lim_{t \to 1} (2t+1) = 3$$

Instantenious velocity and slope of the tangent line are mathematically the same concept.