Lecture 5: Limits at Infinity

Infinity has been perplexing thinkers since immemorial times. What role does infinity play in our understanding of the universe?

Ex. (Arkady's Generosity) I love to make people happy. Let me offer you a prize: I can give you the following sum in Dollars.

$$2^1 + 2^2 + 2^3 + 2^4 + 2^5 + \ldots + 2^n + \ldots +$$

Think of this as an infinity of money piles. The first pile has \$2, the second pile has \$$2^2$, etc. Would you take this prize? Why or why not?

Solution:

Ask yourself, what would you get? Ask mathematically.

Suppose you are getting S Dollars.

$$S = 2 + \underbrace{\left(2^2 + 2^3 + 2^4 + 2^5 + \ldots +\right)}_{\text{Almost } S \ldots \text{ Hmmm}}$$

$$= 2 + 2\underbrace{\left(2 + 2^2 + 2^3 + 2^4 + \ldots +\right)}_{S}$$

So $S = 2 + 2S$

or $S = -2$

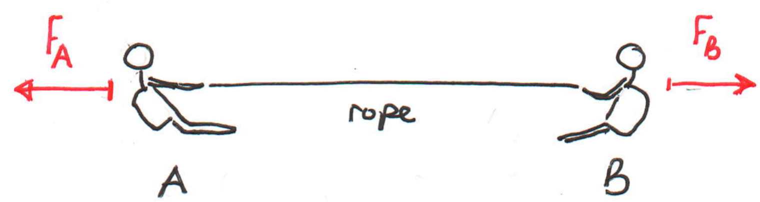
You take my offer and give me \$2 !

You might object, noting that if $S = +\infty$

$+\infty = 2 + 2(\infty)$. However $+\infty$ is growth without bound. It is not a "real" infinity. This explanation may not be satisfying. In physics, it seems, that nature will pick a finite solution over an infinite one when there is a choice.

Ex. Two astronauts are pulling on both sides of a rope of mass $m=0$. Astronout A is stronger than astronout B. what can be said about the forces F_A and F_B, due to astronauts A and B respectively?

Solution:



By Newton's Laws $F_A - F_B = ma$

if $F_A \neq F_B$, for example, if $F_A > F_B$

$F_A - F_B > 0 \implies$ positive net force to the left is acting on a rope of mass $m=0$.

But $0 \cdot a = 0$ if a is any finite number!

For lack of better notation, we have two possibilities:

1. $0 < F_A - F_B = 0 \cdot \infty$

2. $0 = F_A - F_B = 0 \cdot a$.

Solution 2 is accepted as physically valid. That is the astronauts can never exert unequal forces on the rope.

Infinite sums play surprisingly frequent roles in our day to day. First, let us develop a convenient notation:

$$\sum_{k=1}^{\infty} a_k = a_1 + a_2 + a_3 + \ldots +$$

Ex. Write $\frac{1}{10^1} + \frac{1}{10^2} + \frac{1}{10^3} + \ldots +$ in $\sum$ notation.

Solution:

$$\frac{1}{10^1} + \frac{1}{10^2} + \frac{1}{10^3} + \frac{1}{10^4} + \ldots + = \sum_{k=1}^{\infty} \frac{1}{10^k}.$$

Remark: $\sum_{k=1}^{n} a_k = a_1 + a_2 + a_3 + \ldots + a_n$ is called sigma notation. We will learn much more about it later in the course.

Ex. When you use a calculator to compute $\frac{1}{3}$, the output is 0.3333... What does it mean?

Solution:

$$0.3333\ldots = \frac{3}{10} + \frac{3}{10^2} + \frac{3}{10^3} + \frac{3}{10^4} + \ldots +$$

$$= \sum_{k=1}^{\infty} \frac{3}{10^k}$$

It's an infinite sum! What does it equal?

$$S = \frac{3}{10} + \underbrace{\left(\frac{3}{10^2} + \frac{3}{10^3} + \frac{3}{10^4} + \ldots +\right)}_{\text{Almost } S.}$$

$$S = \frac{3}{10} + \frac{1}{10}\left(\frac{3}{10^{2-1}} + \frac{3}{10^{3-1}} + \frac{3}{10^{4-1}} + \ldots +\right)$$

$$= \frac{3}{10} + \frac{1}{10}\underbrace{\left(\frac{3}{10} + \frac{3}{10^2} + \frac{3}{10^3} + \ldots +\right)}_{\text{Exactly } S.}$$

$$S = \frac{3}{10} + \frac{1}{10} S$$

$$\frac{9}{10} S = \frac{3}{10}$$

$$S = \frac{10}{9} \cdot \frac{3}{10} = \frac{1}{3}.$$

Ex. You roll a fair die until either the number 1 or the number 6 comes up. If number 6 comes before 1, you win. If you roll a 1 before 6, you lose. What is your probability of winning?

Solution:

Let W_k be the event that you win on the k^{th} roll

W_k :	*	*	*		*	$\boxed{6}$
	1	2	3	...	$k-1$	k

* - neither 1 nor 6

$$P(W_k) = \frac{4}{6} \cdot \frac{4}{6} \cdot \frac{4}{6} \cdot \ldots \cdot \frac{4}{6} \cdot \frac{1}{6}$$

$$= \left(\frac{4}{6}\right)^{k-1} \frac{1}{6}$$

The probability that we win is the probability we win on first roll, or on second roll, or on third roll, etc.

$$P = \sum_{k=1}^{\infty} P(W_k) = \sum_{k=1}^{\infty} \left(\frac{4}{6}\right)^{k-1} \frac{1}{6}$$

$$P = \frac{1}{6} + \left(\frac{4}{6}\cdot\frac{1}{6} + \left(\frac{4}{6}\right)^2 \cdot \frac{1}{6} + \left(\frac{4}{6}\right)^3 \cdot \frac{1}{6} + \ldots +\right)$$

$$= \frac{1}{6} + \frac{4}{6}\left(\underbrace{\frac{1}{6} + \frac{4}{6}\cdot\frac{1}{6} + \left(\frac{4}{6}\right)^2 \cdot \frac{1}{6} + \ldots +}_{P}\right)$$

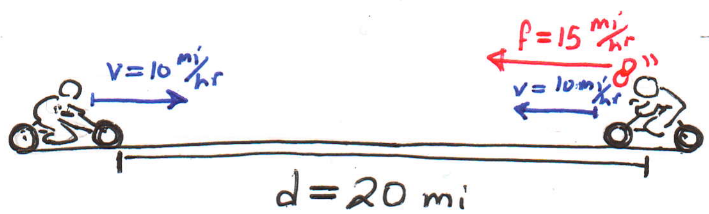
$$= \frac{1}{6} + \frac{4}{6} P$$

$\frac{2}{6} P = \frac{1}{6}$ Hence $P = \frac{1}{2}$.

Does this solution make sence? The game will end in either 1 or 6. By symmetry (since die is fair) Ending in 1 is just as likely as ending in 6.

Thus $P(\text{Ending in } 6) = \frac{1}{2}$.

<u>Ex.</u> Two cyclists start 20 miles apart ride towards each other at 10 mi/hr. A fly sitting on the nose of one cyclist takes off and flies from the nose of one cyclist to the next at 15 mi/hr until the cyclists meet. How much distance does the fly cover?



<u>Solution:</u>

This is a very famous problem that was circulating in the 1950's. There is a clever way to solve this problem instantly. However, when I first saw this problem, I solved it just like the great John von Neumann did.

Von Neumann's solution

Let t_k be the time for trip k of the fly.

Let d_{k-1} be the distance between cyclists as the fly is about to embark on the k^{th} trip.

We have

$$d_{k-1} = (f+v)\,t_k \;;\; d_k = d_{k-1} - 2v t_k$$

Hence $d_k = d_{k-1} - \frac{2v}{f+v}\, d_{k-1} = \left(\frac{f-v}{f+v}\right) d_{k-1}$

$$= \left(\frac{f-v}{f+v}\right)^k d_0$$

Thus $t_k = \dfrac{d_{k-1}}{(f+v)} = \dfrac{1}{(f+v)} \left(\dfrac{f-v}{f+v}\right)^{k-1} d_0$

The fly will undertake infinitely many trips.

The sum of these distances is

$$\sum_{k=1}^{\infty} f t_k = \sum_{k=1}^{\infty} \frac{f d_0}{f+v} \left(\frac{f-v}{f+v}\right)^{k-1}$$

$$= \frac{f d_0}{f+v} \cdot \frac{1}{1 - \frac{f-v}{f+v}}$$

$$= \frac{f d_0}{(f+v)-(f-v)} = \frac{f d_0}{2v} = \frac{15 \cdot 20}{2 \cdot 10}$$

$$= 15$$

The clever solution

Let t be the time it takes the cyclists to meet. All this time the fly would be in the air. Thus, the distance that it will cover is ft. Now $2vt = d$ so $t = \frac{d}{2v}$ and

$$ft = \frac{fd}{2v} = \frac{15 \cdot 20}{2 \cdot 10} = 15$$

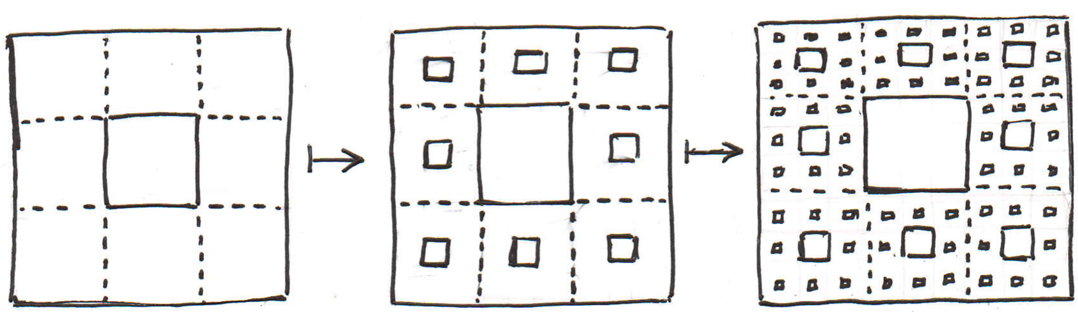
Von Neumann solved this problem instantly. Amazed, the person that asked him this exclaimed:

– Wow! You figured out the trick!

– What trick? I summed a geometric series

Von Neumann replied.

Ex. For best reception, cellphones contain a device called fractal antena. It is manufactured out of a square metal sheet as follows:



Specifically, remove the middle square of area $\frac{1}{9}$. From the remaining 8 subsquares, remove 8 subsquares of size $\left(\frac{1}{9}\right)^2$. Now there are 64 sub-squares. Remove middle sub-sub squares of area $\left(\frac{1}{9}\right)^3$ etc. How much metal do you need to build this antenna?

<u>Solution:</u>

The area of removed metal is

$$A = \frac{1}{9} + \frac{8}{9^2} + \frac{8^2}{9^3} + \frac{8^3}{9^4} + \dots +$$

$$= \frac{1}{9} + \frac{8}{9}\left(\underbrace{\frac{1}{9} + \frac{8}{9^2} + \frac{8^2}{9^3} + \dots +}_{A}\right)$$

$$A = \frac{1}{9} + \frac{8}{9}A$$

$\frac{1}{9}A = \frac{1}{9}$ or $A = 1$. The antenna uses 0 material!

I just gave you a way to make millions from nothing!!!

My brother was present at one of my lectures once. In the evening he told me

– I am interested in infinity. Will you help me?

Excitedly, I said yes! I was so happy that I am such a good lecturer.

The next day he drove me 2 hrs upstate, parked next to an old car and said

– This is a 2001 Infiniti IQ 30. Now drive this car back!

He bought the car for $270. It was in fairly good shape.

Later on he bought a fractal antenna to improve the radio signal.

We will have many more interesting things to say, but first we need to learn a couple of things about limits.

we have studied

$$\lim_{x \to a} f(x)$$

or limits where x approaches real values.

Now let us consider what will happen if x is increased without bound.

$x \to \infty$ means that the values plugged into f are larger and larger.

The meaning of $x \to -\infty$ is similar.

<u>Ex.</u> Calculate $\displaystyle\lim_{x \to \infty} \frac{x^2-1}{x^2+1}$

<u>Solutions</u>

Beginner students like to plug in

$$\frac{\infty^2-1}{\infty^2+1} = \frac{\infty}{\infty} = 1.$$

Well, this is wonderful, except that ∞ is "growth without bound" rather than a number.

We can try this again

$$\lim_{x \to \infty} \frac{x^2-1}{x^2+1} = \lim_{x \to \infty} \frac{\cancel{x^2}\left(1-\frac{1}{x^2}\right)}{\cancel{x^2}\left(1+\frac{1}{x^2}\right)} = \lim_{x \to \infty} \frac{1-\frac{1}{x^2}}{1+\frac{1}{x^2}} = \frac{1-0^+}{1+0^+} = 1.$$

Def: If $\lim\limits_{x \to \infty} f(x) = L$, we say that the line $y = L$ is a horizontal asymptote.

Remark: Think of asymptotes as simple and familiar shapes, which you use to visualize and describe something less familiar.
When you say to a girl: "your eyes shine like two moons" you mean that the moon is an asymptote for each eye. When you look at a cloud and say that it looks like a fluffy sheep, you are using the sheep as an asymptote.

Ex. Find the horizontal asymptotes for $f(x) = \dfrac{2x^3 + x - 1}{x^3 + x^2 + 5}$

Solution:

$$\lim_{x \to \infty} \frac{2x^3 + x - 1}{x^3 + x^2 + 5} = \lim_{x \to \infty} \frac{\cancel{x^3}\left(2 + \frac{x}{x^3} - \frac{1}{x^3}\right)}{\cancel{x^3}\left(1 + \frac{x^2}{x^3} + \frac{5}{x^3}\right)}$$

← factor heighest term (numerator)

← factor heighest term (denominator)

$$= \lim_{x \to \infty} \frac{2 + \frac{1}{x^2} - \frac{1}{x^3}}{1 + \frac{1}{x} + \frac{5}{x^3}} = 2.$$

This means that as we move along the graph $y = f(x)$ to the right, the graph looks more and more like $y=2$. Thus $y=2$ is a horizontal asymptote to the right.

Similarly, since $\lim_{x\to -\infty} f(x) = 2$, $y=2$ is also a horizontal asymptote to the left.

Ex. Since $\lim_{x\to \pm\infty} \frac{1}{x} = 0$, $y=0$ is a horizontal asymptote in both the right and left direction.

There is nothing special about horizontal asymptotes

$y=mx+b$ is a linear asymptote. If $m \neq 0$, we call it an oblique asymptote

We can also have quadratic asymptotes $y=x^2$ cubic asymtotes $y=x^3$ etc. In short, think of the word "asymptote" as a synonim for "familiar".

Ex. Find asymptotes for $f(x) = \dfrac{1+x^6}{1+x^4}$

Solution:

First observe that $\lim_{x\to\pm\infty} \dfrac{x^6+1}{x^4+1} = \lim_{x\to\pm\infty} \dfrac{x^6\left(1+\frac{1}{x^6}\right)}{x^4\left(1+\frac{1}{x^4}\right)}$

$$= \lim_{x\to\pm\infty} x^2 \frac{1+\frac{1}{x^6}}{1+\frac{1}{x^4}} = \infty\cdot 1 = \infty.$$

Using long division, we get

$$\begin{array}{r|l} & x^2 \\ \hline x^4+1 & x^6+1 \\ & -x^6+x^2 \\ \hline & -x^2+1 \end{array}$$

Hence $\dfrac{x^6+1}{x^4+1} = x^2 + \dfrac{-x^2+1}{x^4+1}$

Notice that $\displaystyle\lim_{x \to \pm\infty} \frac{-x^2+1}{x^4+1} = 0$

Thus, for large x, we see less and less difference between $y = \dfrac{x^6+1}{x^4+1}$ and $y = x^2$. So $y = x^2$ is an asymptote.

<u>Ex.</u> What asymptotes does $f(x) = \dfrac{x^5+x^4-4x^3+1}{x^4-4x^3+1}$ have?

<u>Solution:</u> Using long division, we get

$$\begin{array}{r|l} & x+5 \\ \hline x^4-4x^3+1 & x^5+x^4-4x^3+1 \\ & -x^5-4x^4+x \\ \hline & 5x^4-4x^3-x+1 \\ & -5x^4-20x^3+5 \\ \hline & 16x^3-x-4 \end{array}$$

Thus $y = \dfrac{x^5 + x^4 - 4x^3 + 1}{x^4 - 4x^3 + 1} = x + 5 + \dfrac{16x^3 - x - 4}{x^4 - 4x^3 + 1}$

Notice that $\displaystyle\lim_{x \to \pm\infty} \frac{16x^3 - x - 4}{x^4 - 4x^3 + 1} = 0$

Thus $y = x + 5$ is an oblique asymptote.

In particular $y = f(x)$ looks like $y = x + 5$

so $\displaystyle\lim_{x \to \pm\infty} f(x) = \lim_{x \to \pm\infty} (x+5) = \pm\infty.$

Thm: (Think-in-the-box method)

$$\lim_{x \to \pm\infty} \frac{\boxed{a_n x^n} + a_{n-1} x^{n-1} + \dots + a_0}{\boxed{b_m x^m} + b_{m-1} x^{m-1} + \dots + b_0}$$

$$= \lim_{x \to \pm\infty} \frac{a_n x^n}{b_m x^m}$$

Proof: Factor heighest power from numerator and from denominator.

$$\lim_{x \to \pm\infty} \frac{x^n \left(a_n + a_{n-1} \frac{1}{x} + \dots + a_0 \frac{1}{x^n} \right)}{x^m \left(b_m + b_{m-1} \frac{1}{x} + \dots + b_0 \frac{1}{x^m} \right)}$$

$$= \left(\lim_{x \to \pm\infty} \frac{x^n}{x^m} \right) \left(\frac{a_n + a_1 \cdot 0 + \dots + a_0 \cdot 0}{b_m + b_{m-1} \cdot 0 + \dots + b_0 \cdot 0} \right) = \lim_{x \to \pm\infty} \frac{a_n x^n}{b_m x^m}$$

Ex. Use the box method to quickly find the limits at $\pm\infty$.

(a) $$\lim_{x \to \pm\infty} \frac{5x^{10} - 6x^3 + 8x + 10}{10x^{10} + 7x^8 - 16x + 20}$$

(b) $$\lim_{x \to \pm\infty} \frac{6x^3 + 2x - 10}{3x^2 + 10x - 7}$$

(c) $$\lim_{x \to \pm\infty} \frac{-2x^9 + 8x^6 + 7}{5x^{12} - 16x + 9}$$

Solution:

(a) $$\lim_{x \to \pm\infty} \frac{\boxed{5x^{10}} - 6x^3 + 8x + 10}{\boxed{10x^{10}} + 7x^8 - 16x + 20}$$

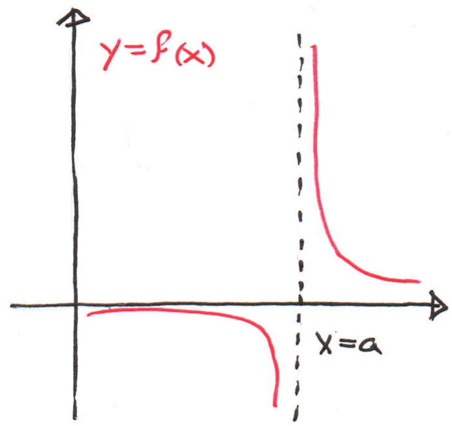
$$= \lim_{x \to \pm\infty} \frac{5x^{10}}{10x^{10}} = \frac{5}{10} = \frac{1}{2}.$$

(b) $$\lim_{x \to \pm\infty} \frac{\boxed{6x^3} + 2x - 10}{\boxed{3x^2} + 10x - 7} = \lim_{x \to \pm\infty} \frac{6x^3}{3x^2} = \pm\infty$$

(c) $$\lim_{x \to \pm\infty} \frac{\boxed{-2x^9} + 8x^6 + 7}{\boxed{5x^{12}} - 16x + 9} = 0.$$

Unlike asymptotes at $\pm\infty$, vertical asymptotes are always vertical lines for functions.

Def: If $\lim_{x \to a^+} f(x) = \pm\infty$ or $\lim_{x \to a^-} f(x) = \pm\infty$ we say that $f(x)$ has a vertical asymptote at $x = a$.



$y = f(x)$ looks like a vertical line near $x = a$.

Ex. Find the asymptotes of $f(x) = \dfrac{\sqrt{2x^2+1}}{3x-5}$

Solution:

Observe that $\lim_{x \to \frac{5}{3}^\pm} \dfrac{\sqrt{2x^2+1}}{3x-5} = \pm\infty$

Hence $x = \frac{5}{3}$ is a vertical asymptote.

$$\lim_{x \to \infty} \frac{\sqrt{2x^2+1}}{3x-5} = \lim_{x \to \infty} \frac{\sqrt{x^2\left(2+\frac{1}{x^2}\right)}}{x\left(3-\frac{5}{x}\right)}$$

$$= \lim_{x \to \infty} \frac{\sqrt{x^2}\sqrt{2+\frac{1}{x^2}}}{x\left(3-\frac{5}{x}\right)} = \lim_{x \to \infty} \frac{|x|\sqrt{2+\frac{1}{x^2}}}{x\left(3-\frac{5}{x}\right)}$$

$$= \lim_{x \to \infty} \frac{\cancel{x}\sqrt{2+\frac{1}{x^2}}}{\cancel{x}\left(3-\frac{5}{x}\right)} = \frac{\sqrt{2}}{3}$$

Thus $y = \frac{\sqrt{2}}{3}$ is a horizontal asymptote to the right of the graph $y = f(x)$.

$$\lim_{x \to -\infty} \frac{\sqrt{2x^2+1}}{3x-5} = \lim_{x \to -\infty} \frac{\sqrt{x^2}\sqrt{2+\frac{1}{x^2}}}{x\left(3-\frac{5}{x}\right)}$$

$$= \lim_{x \to -\infty} \frac{|x|\sqrt{2+\frac{1}{x^2}}}{x\left(3-\frac{5}{x}\right)} = \lim_{x \to -\infty} \frac{-x\sqrt{2+\frac{1}{x^2}}}{x\left(3-\frac{5}{x}\right)}$$

$$= -\frac{\sqrt{2}}{3}$$

Thus $y = -\frac{\sqrt{2}}{3}$ is a horizontal asymptote as we scroll to the left of the graph $y = f(x)$.

<u>Remark:</u> Many students make the mistake of thinking that $\sqrt{x^2} = x$. This is incorrect:

$$\sqrt{(\underbrace{-3}_{x})^2} = \sqrt{9} = 3 = |\underbrace{-3}_{x}|$$

$$\sqrt{x^2} = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Don't assume $-x$ is negative!!!

Ex. Compute $\lim_{x \to +\infty} \left(\sqrt{x^2+1} - x\right)$. What is

$\lim_{x \to -\infty} \left(\sqrt{x^2+1} - x\right)$?

Solution: We will learn better methods later on in the course. Almost every limit that you encounter at this level, which you cannot immediately solve is a derivative in disguise. About that we shall talk again in due time!

For now, let us solve this with algebra.

Some students simply plug ∞ instead of x:

$$\sqrt{(\infty)^2+1} - \infty = \infty - \infty = 0$$

This is wrong, because ∞ is not a number, but rather a state of boundless growth. One '∞' might grow faster than another '∞'.

Instead

$$\lim_{x \to \infty} \frac{\left(\sqrt{x^2+1} - x\right)\left(\sqrt{x^2+1} + x\right)}{\left(\sqrt{x^2+1} + x\right)}$$

$$\lim_{x \to \infty} \frac{(x^2+1) - x^2}{\sqrt{x^2+1} + x} = \lim_{x \to \infty} \frac{1}{\sqrt{x^2+1} + x} = 0$$

On the other hand, $\lim_{x \to -\infty} \sqrt{x^2+1} - x$

$$= \sqrt{(-\infty)^2+1} - (-\infty) = \infty + \infty = \infty$$

$\infty + \infty$ is unambiguous.

<u>Ex.</u> Compute $\lim_{x \to \infty} \left(\sqrt{4x^2 - 3x} - 2x\right)$

<u>Solution:</u>

Just to tease you, I can tell this limit is $\frac{-3}{4}$ at a glance.

$$\lim_{x \to \infty} \frac{\left(\sqrt{4x^2-3x} - 2x\right)\left(\sqrt{4x^2-3x} + 2x\right)}{\left(\sqrt{4x^2-3x} + 2x\right)}$$

$$= \lim_{x \to \infty} \frac{4x^2 - 3x - 4x^2}{\sqrt{4x^2-3x} + 2x} = \lim_{x \to \infty} \frac{-3\cancel{x}}{\cancel{x}\left(\sqrt{4 - \frac{3}{x}} + 2\right)}$$

$$= \frac{-3}{\sqrt{4} + 2} = \frac{-3}{4}$$

<u>Ex.</u> Can you calculate $\lim\limits_{x \to \infty} \left(\sqrt[3]{8x^3 - 5x^2} - 2x \right)$?

<u>Solution:</u>

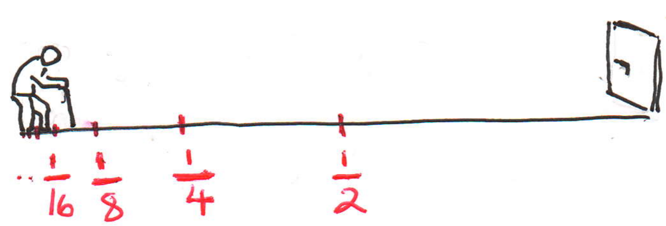
I see the solution instantly as $\frac{-5}{12}$.

Think! Don't memorize! You will be able to do that too. Promise.

Infinity and Paradox

Perhaps the most famous paradox is Zeno's proof for the impossibility of motion.

According to Zeno, you cannot stand up and leave the room, because to do that, you will have to walk ½ the distance to the door, and to do that, you must walk ¼ the distance and so on. You will have to take infinitely many steps in a moment!



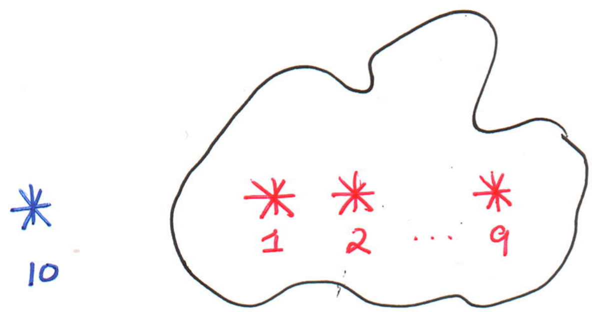
Do you agree with Zeno? Let's see.

<u>Ex.</u> (Universe creation Experiment)

<u>Experiment 1</u>

Stars 1, 2, 3, …, will come into existence as follows.

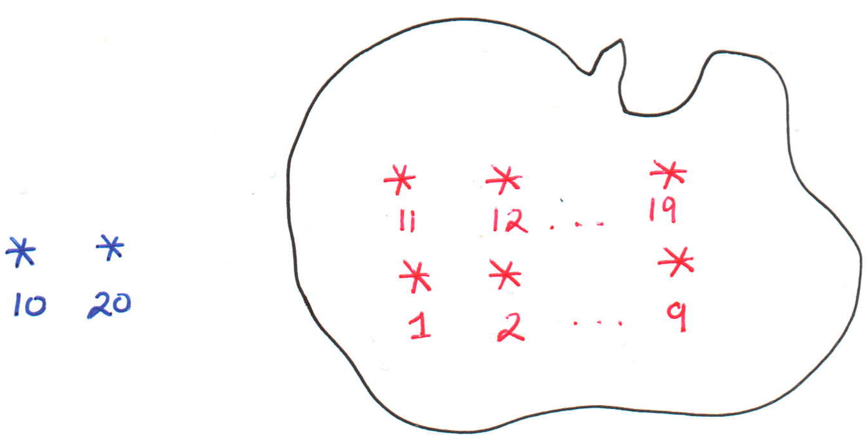
<u>step 1 :</u> $\left(\frac{1}{2}\right)^1$ mins before 12:00, stars 1–10 come to existence, star 10 immediately expires.



Universe

stars = 9·1

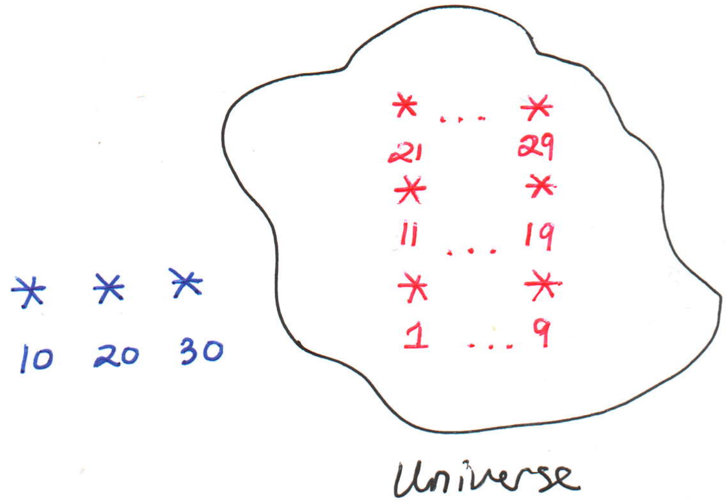
<u>Step 2 :</u> $\left(\frac{1}{2}\right)^2$ mins before 12:00, stars 11–20 come to existence, star 20 immediately expires.



Universe

stars = 9·2

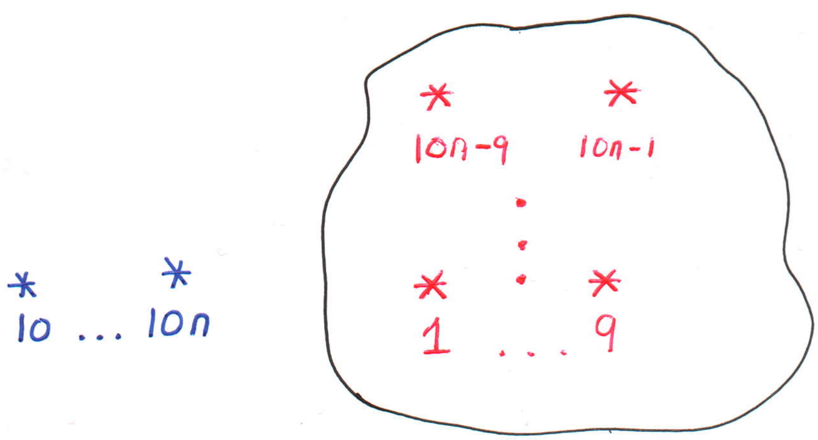
<u>Step 3:</u> $\left(\frac{1}{2}\right)^3$ mins before 12:00, stars 21-30 come into existence. Star 30 is immediately extinguished.



stars = 9·3

⋮

<u>Step n:</u> $\left(\frac{1}{2}\right)^n$ mins before 12:00, stars $10 \cdot n-9$ - $10n$ come into existence, star $10n$ is immediately extinguished.



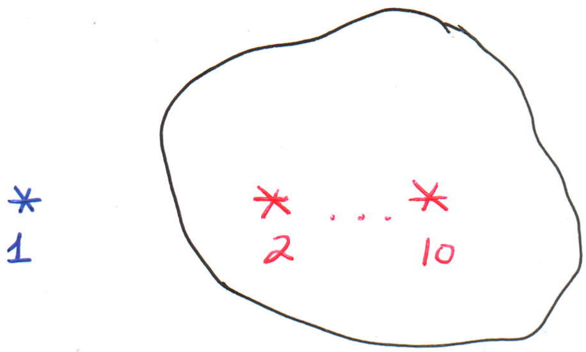
stars = $9 \cdot n$

<u>Q.1</u> How many stars in the universe at 12:00?

We will wait with the answer until after the second experiment will have been explained.

Experiment 2

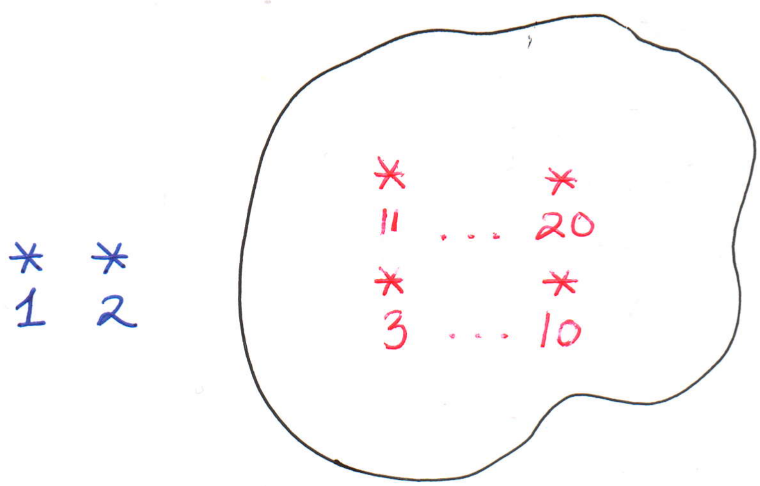
Step 1: $\left(\frac{1}{2}\right)^1$ mins before 12:00, stars 1-10 come into existence. Star 1 is immediately extinguished.



Universe

stars = 9·1

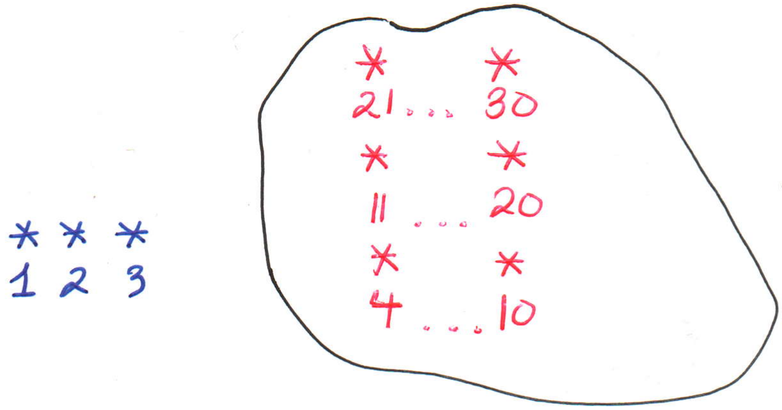
Step 2: $\left(\frac{1}{2}\right)^2$ mins before 12:00, stars 11-20 come into existence. Star 2 is immediately extinguished



Universe

stars = 9·2

<u>Step 3 :</u> $\left(\frac{1}{2}\right)^3$ mins before 12:00, stars 21-30 come into existence. Star 3 is immediately extinguished.

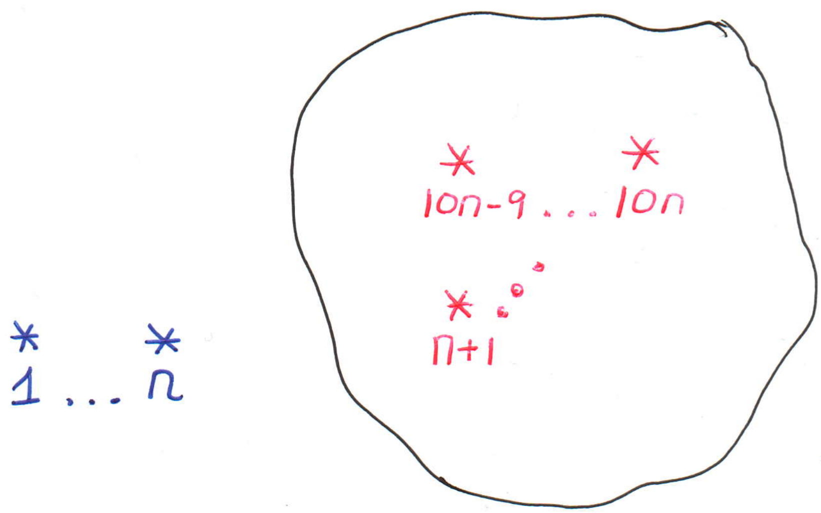


stars $= 9 \cdot 3$

Universe

⋮

<u>step n :</u> $\left(\frac{1}{2}\right)^n$ mins before 12:00, stars $10n-9$ – $10n$ come into existence, Star n is immediately extinguished



stars $= 9 \cdot n$

<u>Q. 2</u> How many stars in the universe at 12:00?

A1: The number of stars at 12:00 is infinite.

Stars numbered 10, 20, 30, or any star whose number is divisible by 10 vanish from the universe.

All other stars remain there.

Note that $\lim_{n \to \infty} 9 \cdot n = \infty$

A2: Does any star survive in the universe?

Well, star 1 expires in step 1, star 2 expires in step 2, in general star n expires in step n.

Even though the number of stars grows without bound as time approaches 12 $\left(\lim_{n \to \infty} 9 \cdot n = \infty\right)$, at 12:00 the universe is dark and empty.

Remark: A frequent argument that I hear is that time will never reach 12. But does time care how it is being used?

In the words of the great Bulat Okudzava

А время торопит возница беспечный

И просятся кони вперёд.

And time is rushed by indifferent driver
And horses ask to move forward.

Logically, if you belive that time will not reach 12:00 then you must also belive that walking out of the room is also impossible; if you are walking towards the door at constant speed, you will have to carry out infinitely many steps. For instance if the door is 1 meter away from you and you move 1 meter/min

<u>step 1:</u> $\left(\frac{1}{2}\right)^1$ - mins before 12:00 walk half the distance.

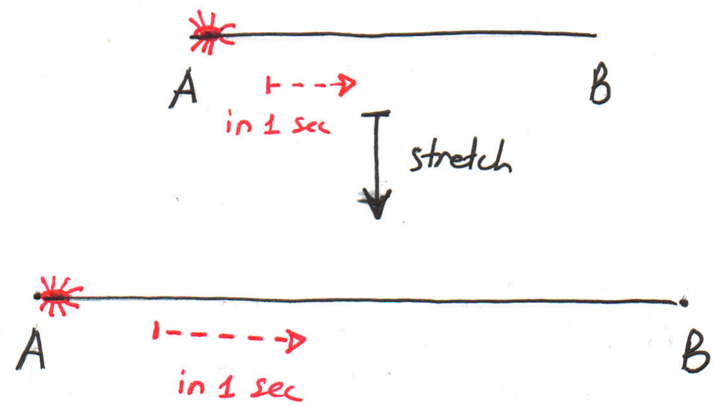
<u>Step 2:</u> $\left(\frac{1}{2}\right)^2$ - mins before 12:00 walk 1/4 the distance

⋮

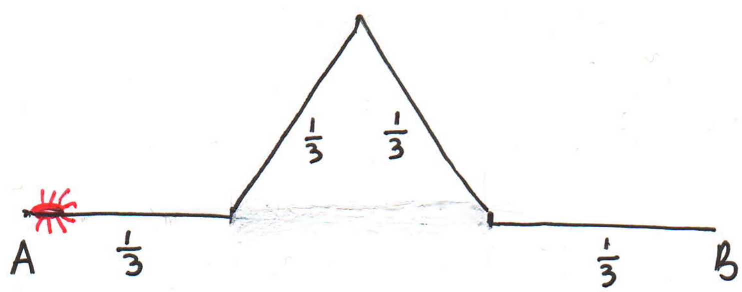
<u>step n:</u> $\left(\frac{1}{2}\right)^n$ - mins before 12:00 walk $\left(\frac{1}{2}\right)^n$ the distance.

Can you ever get from point A to point B? If so, do you ever consider it miraculus?

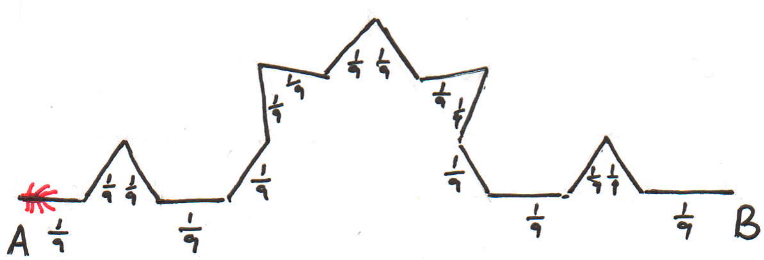
Imagine that you are watching a video of a bug moving on a flexible screen. The bug moves from point A to point B in 1 sec.

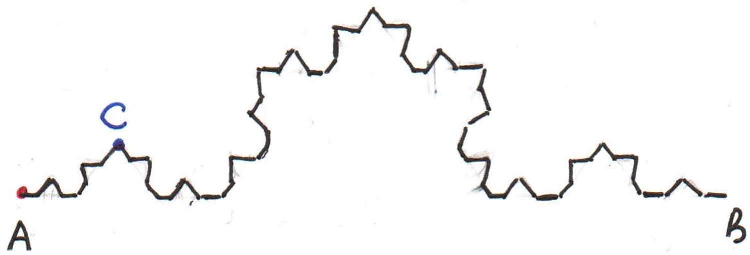


Step 1:



Step 2:





Deform the sreen as shown in the pictures.

<u>step 1</u>: Deform the middle thirds of the screen to twice the size. length of screen $= \frac{4}{3}$

<u>Step 2</u>: Divide each of the 4 segments of screen from step 1 further into 3 parts. Elongate the middle portion to twice the size. length of screen $= \left(\frac{4}{3}\right)^2$

<u>Step 3</u>: Divide each of the 16 segments of screen from step 2 into 3 parts. Elongate the middle portion to twice the size. length of screen $= \left(\frac{4}{3}\right)^3$

⋮

<u>step n</u>: Divide each of the 4^n segments of screen into 3 parts. Elongate the middle portion to twice

the size. length of screen = $\left(\frac{4}{3}\right)^n$.

As $\lim_{n \to \infty}$ step n goes to infinity, we generate the famous von Koch curve.

Q. How long do you think is the path from A to B?

A. As viewed from the screen, the path is infinitely long.

Q. How far is any point on the curve from A?

A. Within the one dimensional space of the von Koch curve, the distance is infinite.

I am this bug ladies and gentlemen.