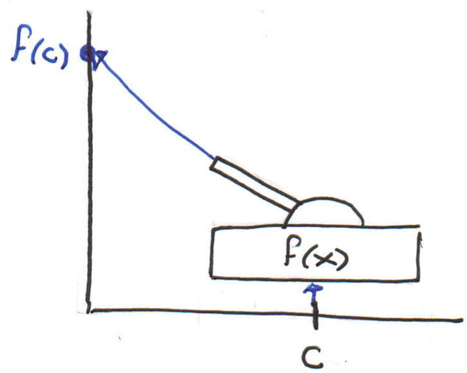
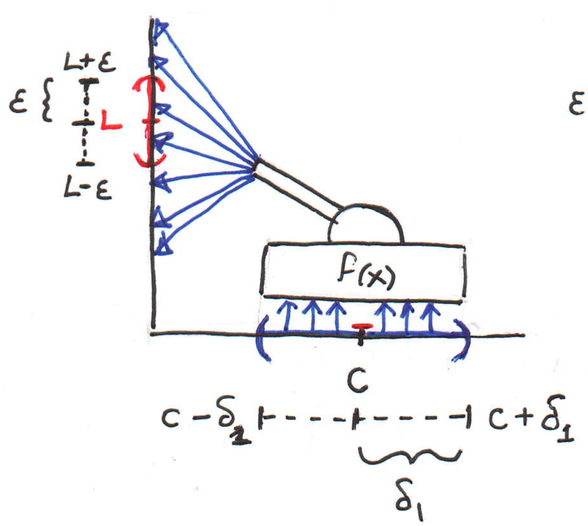
Lecture 3: The Precise Definition of Limit

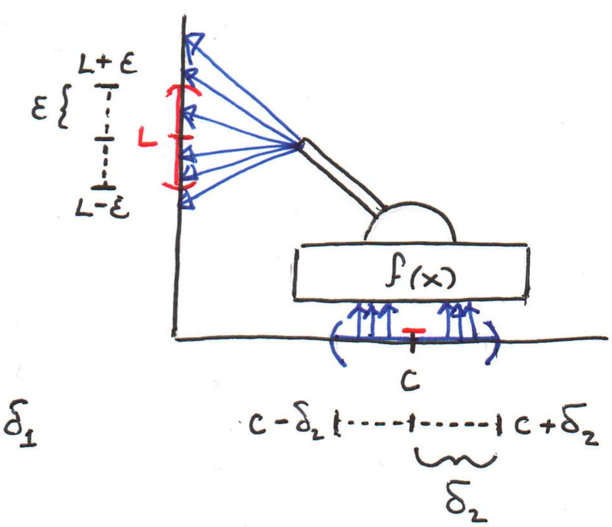
We have already observed that $f(c)$ and $\lim_{x \to c} f(x)$ are very different operations. A useful analogy is a water canon on the x-axis.

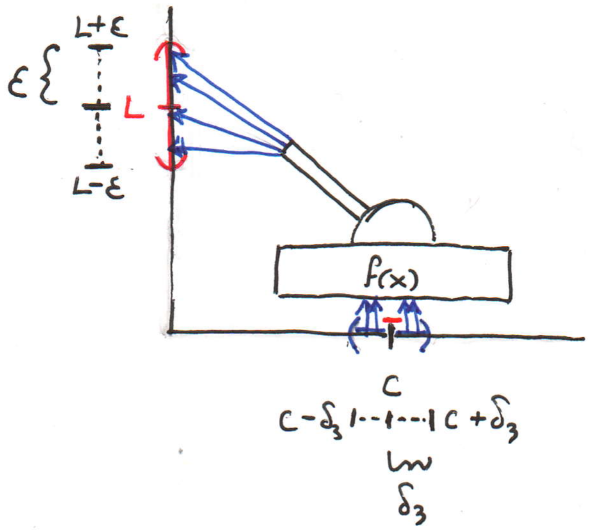


Evaluation of $f(x)$ at c corresponds to shooting the particle c from the x-axis onto the y-axis. $f(c)$ is the point where the projectile hits.

The limit process is a bit more complicated:

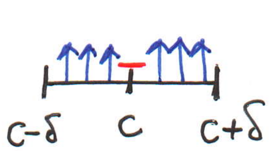




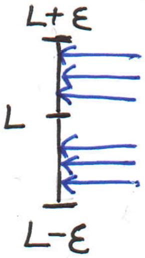


We say that $\lim_{x \to c} f(x) = L$ whenever the current of particles lausched from the x-axis near c can be focused about L by narrowing the pipe about c.

More precisely, for any pipe on the y-axis of radius ε, we can select a pipe on the x-axis of radius δ, such that all the particles coming from the x-pipe



$\Longleftrightarrow \{x : |x-c| < \delta\}$ land into the y-pipe



$\Longleftrightarrow \{y : |y-L| < \varepsilon\}$.

<u>Def:</u> We say that $\lim_{x \to c} f(x) = L$ if for every $\varepsilon > 0$, there is a number $\delta(\varepsilon) > 0$ such that

$$|f(x) - L| < \varepsilon$$

whenever $0 < |x - c| < \delta(\varepsilon)$.

Again, the definition means:
We can get all points near c to be mapped near L, where proximity to L is measured by ε and the proximity to c is measured by δ.
$\delta(\varepsilon)$ is a function that specifies how much near c all x must be so that $f(x)$ gets into the pipe of radius ε about L.

<u>Ex.</u> Use δ-ε argument to show that

$$\lim_{x \to 2} (3x - 1) = 5$$

<u>Solution:</u> We must show that for any $\varepsilon > 0$ we can find $\delta(\varepsilon) > 0$ such that

$$|3x - 1 - 5| < \varepsilon$$

whenever $\quad 0 < |x-2| < \delta(\varepsilon)$.

We think in reverse:

Suppose we picked the right δ-pipe and achieved

$$|3x-1-5| < \varepsilon$$

what was δ ?

$$\varepsilon > |3x-6| = |3(x-2)| = 3|x-2|$$

so $\quad |x-2| < \frac{\varepsilon}{3}$

But $|x-2|$ is the distance of the approximant x to 2.

Hence $\quad |x-2| < \frac{\varepsilon}{3} = \delta(\varepsilon)$.

<u>Remark</u>: $\delta(\varepsilon) = \frac{\varepsilon}{3}$ means that the radius of the input pipe δ must be at least $\frac{1}{3}$ of the radius of the target pipe ε.

For example, if $\varepsilon = 1$, $\delta = \frac{1}{3}$, if $\varepsilon = \frac{1}{2}$, $\delta = \frac{1}{6}$ etc.

Ex. Use δ-ε argument to show that

$$\lim_{x \to -2} (5x+2) = -8$$

Solution:

$$\varepsilon > |5x+2-(-8)| = |5x+10|$$

$$= 5|x+2|$$

so $|x+2| < \frac{\varepsilon}{5}$ or $\delta(\varepsilon) = \frac{\varepsilon}{5}$.

Ex. Use δ-ε argument to show that

$$\lim_{x \to a} (mx+b) = ma+b$$

Solution:

$$\varepsilon > |(mx+b)-(ma+b)| = |m(x-a)| = |m||x-a|$$

Hence $|x-a| < \frac{\varepsilon}{|m|} = \delta(\varepsilon)$.

$\delta-\varepsilon$ arguments are not always so straight forward

Ex. Use $\delta-\varepsilon$ argument to show that

$$\lim_{x \to 4} x^2 = 16$$

Solution:

Many students begin as before

$$\varepsilon > |x^2-16| = |(x+4)(x-4)| = |x+4||x-4|$$

Thereupon the answer given is

$$\delta(\varepsilon) = \frac{\varepsilon}{|x+4|} > |x-4|.$$

Q. What is wrong?

A. Remember what is it that we want to accomplish? For a given ε, we want to build a pipe of a fixed radius δ. But the width of the pipe fluctuates with x if we have $\delta = \frac{\varepsilon}{|x+4|}$.

Do you see that? If $x = 4.01$, $\delta = \frac{\varepsilon}{8.01}$ and if $x = 4.001$, $\delta = \frac{\varepsilon}{8.001}$.

Instead, we may think as follows:

Since $x \to 4$ we are making δ small anyways.

Lets consider only pipes of radius $\delta \leq 1$.

So $\quad |x-4| \leq 1 \quad$ or

$$-1 \leq x-4 \leq 1$$

$$4-1 \leq x \leq 4+1$$

$$3 \leq x \leq 5$$

$$4+3 \leq x+4 \leq 4+5$$

$$7 \leq x+4 \leq 9$$

$$|x+4| \leq 9.$$

Hence $\quad |x^2-16| = |x+4||x-4| \overset{\delta \leq 1}{\leq} 9|x-4| \overset{\delta < \frac{\varepsilon}{9}}{<} \varepsilon$

Thus we may set $\delta(\varepsilon) = \min\left\{1, \frac{\varepsilon}{9}\right\}$

Do you see what I mean?

If $\quad |x-4| < \delta(\varepsilon) \quad$ then simultaneously

① $\quad |x-4| < \dfrac{\varepsilon}{9}$

$$9|x-4| < \varepsilon$$

② $\quad |x-4| < 1$

$$|x+4||x-4| \leq 9|x-4|$$

This makes the inequalities

$$|x^2-16| = |x+4||x-4| \underset{(2)}{\leqslant} 9|x-4| \underset{(1)}{<} \varepsilon$$

<u>Ex.</u> Show using a δ-ε argument that

$$\lim_{x \to -1} 2x^2-5x-2 = 5.$$

<u>Solution:</u>

From $|(2x^2-5x-2)-5| < \varepsilon$ we want to extract information about $|x+1|$.

$$|2x^2-5x-7| = |x+1||2x-7|$$

If $|x+1| < 1$

$$-1 < x+1 < -1$$

$$-2 < x < 0$$

$$-11 < 2x-7 < -7$$

$$|2x-7| < 11$$

So $|x+1||2x-7| \underset{\delta<1}{<} 11|x+1| \underset{\delta<\frac{\varepsilon}{11}}{<} \varepsilon$

Thus $\delta(\varepsilon) = \min\{1, \varepsilon/11\}$

Ex. Use δ-ε argument to show that

$$\lim_{x \to t} ax^2 + bx + c = at^2 + bt + c.$$

Solution: This shows that the limit of a quadratic polynomial at any value t is the same as plugging in that value.

$$|(ax^2+bx+c) - (at^2+bt+c)| =$$

$$= |a(x^2-t^2) + b(x-t)| =$$

$$= |a(x-t)(x+t) + b(x-t)| =$$

$$= |a(x+t)+b|\,|x-t| < |a|(|x+t| + |b|)|x-t|$$

Set $\quad |x-t| < 1$

$$-1 < x-t < 1$$

$$2t-1 < x+t < 2t+1$$

$$|x+t| < \max\{|2t-1|, |2t+1|\} = M$$

$$|a|(|x+t| + |b|) < |a|(M+|b|)$$

Thus we may pick

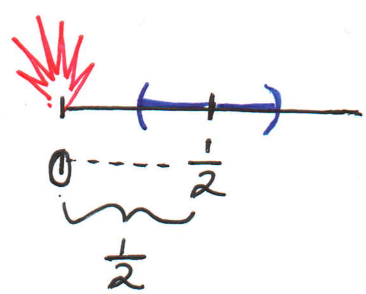
$$\delta(\varepsilon) = \min\left\{1, \ \varepsilon/|a|(M+|b|)\right\}$$

you might be wondering why we were always making $\delta < 1$. This was arbitrary, we could have picked $\delta < 3$, $\delta < 1/2$, $\delta < \frac{1}{100}$. In a limit $\lim_{x \to c} f(x)$, δ inevitably gets to be small.

<u>Ex.</u> Use δ-ε argument to prove

$$\lim_{x \to \frac{1}{2}} \frac{1}{x} = 2$$

Here we cannot pick δ values $\geq \frac{1}{2}$, because we must avoid feeding 0 into $\frac{1}{x}$.



Set $\delta \leq \frac{1}{4}$

$$\left|\frac{1}{x}-2\right| = \left|\frac{1-2x}{x}\right| = \left|\frac{-2}{x}\left(x-\frac{1}{2}\right)\right|$$

$$= \frac{2}{|x|}\left|x-\frac{1}{2}\right|$$

Now, $\left|x-\frac{1}{2}\right| < \frac{1}{4}$

$$-\frac{1}{4} < x - \frac{1}{2} < \frac{1}{4}$$

$$\frac{1}{4} = \frac{1}{2} - \frac{1}{4} < x < \frac{1}{4} + \frac{1}{2} = \frac{3}{4}$$

$$4 > \frac{1}{x} > \frac{4}{3}$$

$$8 > \frac{2}{x} > \frac{8}{3}$$

$$8 > \left|\frac{2}{x}\right|$$

Thus $\frac{2}{|x|}\left|x-\frac{1}{2}\right| < 8\left|x-\frac{1}{2}\right| < \varepsilon$

if $\left|x-\frac{1}{2}\right| < \frac{1}{4}$ if $\left|x-\frac{1}{2}\right| < \frac{\varepsilon}{8}$

And therefore $\delta(\varepsilon) = \min\left\{\frac{1}{4}, \frac{\varepsilon}{8}\right\}$ works.

All the excersises we covered so far were only useful to explain how δ-ε argument is supposed to work. The first real application is to establish the limit laws.

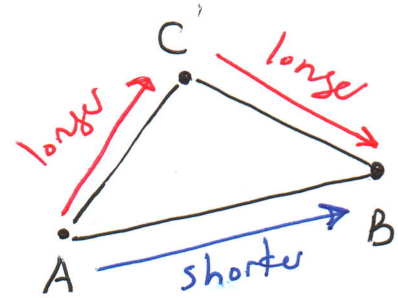
<u>Ex.</u> Use δ-ε argument to show that if

$\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = M$ then

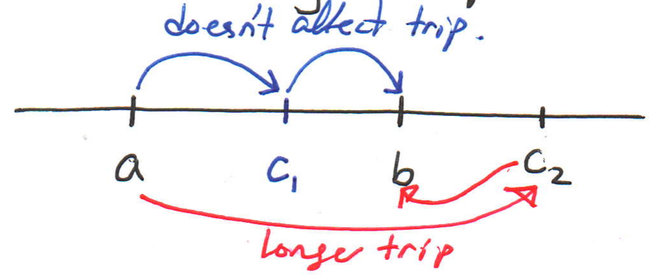
$$\lim_{x \to a} \big(f(x)+g(x)\big) = L+M.$$

<u>Solution:</u> A useful inequality in mathematics is called triangle inequality. In higher dimensions it says that the shortest path from one corner of a triangle to another is along the edge. You may also paraphrase it as:

The shortest path between two points is a line.



In 1-D, distance between points a and b is $|b-a|$ visiting point c might take you out of your way if c is not along the path from a to b.



The illustration shows that

$$|b-a| = |c_1 - a| + |b - c_1|$$

but $\quad |b-a| < |c_2 - a| + |b - c_2|.$

At any rate, for any a, b, c

$$|b-a| \leqslant |c-a| + |b-c|.$$

This can be proved rigorously:

It is enough to pick $c=0$. and show that

$$|b-a| \leqslant |b| + |a|.$$

Observe that $|b \pm a|^2 = (b \pm a)^2 = b^2 \pm 2ab + a^2$

$$\leqslant |b|^2 + 2|a||b| + |a|^2 = (|b| + |a|)^2$$

Thus $|b-a| \leqslant \sqrt{(|b|+|a|)^2} = |b| + |a|.$

For any c

$$|b-a| = |(b-c)+(c-a)| \leqslant |b-c| + |c-a|.$$

Now back to the problem.

We know that $\lim_{x \to a} f(x) = L$ so there exists $\delta_f(\varepsilon)$ s.t. $|f(x) - L| < \varepsilon$ whenever $0 < |x-a| < \delta_f(\varepsilon).$

Similarly, $\lim_{x\to a} g(x) = M$ implies that there exists $\delta_g(\varepsilon)$ s.t. $|g(x) - M| < \varepsilon$ whenever $0 < |x-a| < \delta_g(\varepsilon)$.

$$|(f(x)+g(x)) - (L+M)| = |(f(x)-L) + (g(x)-M)|$$

$$\leq |f(x) - L| + |g(x) - M|.$$

Now if $\delta \leq \delta_f(\varepsilon/2)$, $|f(x) - L| < \varepsilon/2$

and if $\delta \leq \delta_g(\varepsilon/2)$, $|g(x) - M| < \varepsilon/2$

so pick $\delta_{f+g}(\varepsilon) = \min\{\delta_f(\varepsilon/2), \delta_g(\varepsilon/2)\}$.

<u>Ex.</u> Use δ-ε argument to prove that if $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$ then

$$\lim_{x\to a} f(x)g(x) = LM$$

<u>Solution:</u>

Let $\delta_f(\varepsilon)$, $\delta_g(\varepsilon)$ be as in the example above.

$$|f(x)g(x) - LM| = |f(x)g(x) - Lg(x) + Lg(x) - LM|$$

$$\leq |f(x)g(x) - Lg(x)| + |Lg(x) - LM|$$

$$= |g(x)||f(x)-L| + |L||g(x)-M| =$$

$$= |g(x) - M + M||f(x)-L| + |L||g(x)-M|$$

$$\leq \left(|g(x)-M| + |M|\right)|f(x)-L| + |L||g(x)-M|$$

$$= |f(x)-L||g(x)-M| + |M||f(x)-L| + |L||g(x)-M|$$

Since our goal is to focus the current at LM, we may insist that $\varepsilon < 1$.

Setting $\delta(\varepsilon) = \min\left\{ \delta_f\left(\frac{\varepsilon}{1+|L|+|M|}\right), \delta_g\left(\frac{\varepsilon}{1+|L|+|M|}\right) \right\}$

gives

$$|f(x)-L||g(x)-M| + |M||f(x)-L| + |L||g(x)-M|$$

$$< \frac{\varepsilon^2}{(1+|L|+|M|)^2} + \frac{|M|\varepsilon}{1+|L|+|M|} + \frac{|L|\varepsilon}{1+|L|+|M|}$$

$$< \frac{\varepsilon + |M|\varepsilon + |L|\varepsilon}{1+|L|+|M|} = \varepsilon.$$

Comprehension Check

Suppose $\lim\limits_{x\to 5} f(x) = 10$ and use δ-ε argument to show that $\lim\limits_{x \to 5} 2f(x) = 20$

<u>Solution:</u> There exists by hypothesis a function $\delta_f(\varepsilon)$ s.t. $|f(x) - 10| < \varepsilon$ whenever

$0 < |x-5| < \delta_f(\varepsilon)$

Clearly $|2f(x) - 20| < \varepsilon$ iff $|f(x) - 10| < \frac{\varepsilon}{2}$

Hence $\delta(\varepsilon) = \delta_f(\varepsilon/2)$ works.

Comprehension Check

Suppose $\lim\limits_{x \to 3} f(x) = 7$ and $\lim\limits_{x \to 7} g(x) = 1$

Show using $\delta - \varepsilon$ argument that

$$\lim_{x \to 3} g(f(x)) = 1$$

<u>Solution:</u> By hypothesis there is $\delta_g(\varepsilon)$ s.t. $|g(x) - 1| < \varepsilon$ whenever $0 < |x-7| < \delta_g(\varepsilon)$. and $\delta_f(\varepsilon)$ s.t. $|f(x) - 7| < \varepsilon$ whenever $0 < |x-3| < \delta_f(\varepsilon)$.

Let $\delta(\varepsilon) = \delta_f(\delta_g(\varepsilon))$. Then

$$|x-3| < \delta(\varepsilon) \Rightarrow |f(x) - 7| < \delta_g(\varepsilon) \Rightarrow |g(f(x)) - 1| < \varepsilon.$$